

Small resolutions of minuscule Schubert varieties

NICOLAS PERRIN

Abstract

Let X be a minuscule Schubert variety. In this article, we use the combinatorics of quivers to define new quasi-resolutions of X . We describe in particular all relative minimal models $\hat{\pi} : \hat{X} \rightarrow X$ of X and prove that all the morphisms $\hat{\pi}$ are small (in the sense of intersection cohomology). In particular, all small resolutions of X are given by the smooth relative minimal models \hat{X} and we describe all of them. As another application of this description of relative minimal models, we give a more intrinsic statement of the main result of [Pe1].

INTRODUCTION

Schubert varieties have been intensively studied and are of great importance in representation theory. There are several ways to understand the geometry and the singularities of Schubert varieties. One way is to describe the singular locus, its irreducible components and the generic singularity in each of these components. This has been completely achieved for GL_n only recently (see [Man1], [Man2], [BW], [KLR] and [Cor2]). For the general case, there are only partial results (for an account, see [BL]). This description of singularities enables in particular to calculate the Kazhdan-Lusztig polynomials. For this study, combinatoric tools are useful but it is also useful to construct special resolutions of Schubert varieties (see for example [Cor1] and [Cor2]). Another way to study Schubert varieties is to calculate the cohomology of line bundles and especially to prove some vanishing theorems. This has been done in many ways, one of them thanks to the Bott-Samelson resolutions ([Ke], [MR], [RR] or [Ra]). In this article, we want to study the geometry of Schubert varieties thanks to the study of some particular resolutions of this variety.

A nice resolution of Schubert varieties is the Bott-Samelson resolution (see [De] or [Ha]). This resolution is useful to study the singularities and standard monomial theory of Schubert varieties (see [LLM]) and also to study the geometry of Schubert varieties (for curves on minuscule Schubert varieties, see for example [Pe1] and [Pe4]). However these resolutions are big in the sense that the fibers have big dimensions and there are many contracted subvarieties. Another

class of resolutions is of particular importance, the IH -small resolution (that is to say the small resolution in the sense of Intersection Cohomology, see definition 7.1). These are well suited for the calculation of Kazhdan-Lusztig polynomials. In particular A. Zelevinsky in [Ze] constructed some IH -small resolutions for grassmannian Schubert varieties and gave a geometric interpretation of the combinatoric computation of Kazhdan-Lusztig polynomials by A. Lascoux and M.-P. Schützenberger in [LS]. Later P. Sankaran and P. Vanchinathan [SV1] and [SV2] constructed small resolution of some minuscule and cominuscule Schubert varieties for SO_{2n} and Sp_{2n} and calculated the corresponding Kazhdan-Lusztig polynomials. In their article [SV1], they construct some minuscule Schubert varieties not admitting small resolutions. These examples are locally factorial Schubert variety (more precisely with singularities in codimension 2) for which the theorem of purity (see [Gr] theorem 21.12.12) says that there is no IH -small resolution.

In this article we study the IH -small resolutions of minuscule Schubert varieties. We will generalise the constructions of A. Zelevinsky and P. Sankaran and P. Vanchinathan to any minuscule Schubert variety and describe all IH -small resolutions. We do not address the problem of calculating Kazhdan-Lusztig polynomials. This has been done in a combinatoric way for all minuscule Schubert varieties by B. D. Boe in [Bo] and we hope that our construction will lead to a geometric interpretation of these results.

In order to define our resolutions, we introduce a combinatorial object: a quiver associated to a minuscule Schubert variety $X(w)$. This quiver is defined thanks to a reduced writing \tilde{w} of w which is unique for minuscule Schubert variety. These quivers seem to be the same as the quivers defined by S. Zelikson in [Zel] but we did not check this. To this quiver, we associate a configuration variety $\tilde{X}(\tilde{w})$ which is simply the Bott-Samelson resolution. The fact that the Bott-Samelson resolution can be seen as a configuration variety was already known by P. Magyar [Ma] but the use of the quiver in the situation is very useful. In the minuscule case, the quiver will have a very special and rigid geometry and in particular we will define the pics of the quiver and the height of a pic. As in the case of A. Zelevinsky's construction [Ze], the choice of an order on the pics will lead to a partial resolution $\hat{X}(\hat{w}) \rightarrow X(w)$ of the Schubert variety. However, the variety $\hat{X}(\hat{w})$ will in general be locally factorial but not smooth.

These varieties are however interesting for the relative minimal model program. We study the relative minimal models of $X(w)$ and prove the following theorem:

THEOREM 0.1. — *The relative minimal models of $X(w)$ are the varieties $\hat{X}(\hat{w})$ obtained thanks to an order on the pics preserving the order on the heights of the pics (see construction 3 for more details).*

For grassmannian Schubert varieties, there are always IH -small resolution [Ze]. As P. Sankaran and P. Vanchinathan proved this is not true for a general minuscule Schubert variety. The following theorem proves that we need to replace IH -small resolutions by relative minimal models to generalise the result of A. Zelevinsky to any minuscule Schubert variety (that is to say allow locally factorial singularities):

THEOREM 0.2. — *The morphism $\hat{\pi} : \hat{X}(\hat{w}) \rightarrow X(w)$ from a relative minimal model to a minuscule Schubert variety $X(w)$ is small.*

In other words, the relative minimal models play the role for general minuscule Schubert varieties of IH -small resolution for grassmannian Schubert varieties. However they do not share the same nice properties and in particular are not as well fitted as IH -small resolutions for the computation of Kazhdan-Lusztig polynomials. We also describe all relative canonical models of $X(w)$. Furthermore, the following result of B. Totaro [To] using a key result of J. Wisniewski [Wi] tells us to look for IH -small resolutions in the class of relative minimal models:

THEOREM 0.3. — *[Totaro-Wisniewski] Any IH -small resolution of a normal variety X is a relative minimal model for X .*

In particular, in our situation, all IH -small resolutions of $X(w)$ are given by the smooth relative minimal models. We then give a combinatorial criterion on the quiver for the relative minimal model $\hat{X}(\hat{w})$ to be smooth. We get the following:

THEOREM 0.4. — *The variety $\hat{X}(\hat{w})$ is an IH -small resolution if and only if the order on the pics preserves the order on the heights of the pics and if at step i the pic p_i is minuscule for the quiver (see definition 7.11).*

In particular, we are able to say which minuscule Schubert variety admits an IH -small resolution. At the end of the article we sketch another way to prove this: an IH -small resolution has to factor through the relative canonical minimal model $X_{can}(w)$ of $X(w)$ and is a crepant resolution of $X_{can}(w)$. In particular, the stringy Euler number e_{st} defined by V. Batyrev [Ba] of $X_{can}(w)$ has to be an integer. We give a formula for $e_{st}(X_{can}(w))$ and it is an easy verification that we recover in this way all minuscule Schubert varieties not admitting an IH -small resolution.

Another motivation for the study of resolutions and partial resolutions of Schubert varieties (and in fact our motivation at the beginning of the study) is the following reinterpretation of our result in [Pe1]. Let X be a minuscule Schubert variety and $\pi : \hat{X} \rightarrow X$ any relative minimal model. For a 1-cycle class $\alpha \in A_1(X)$ define the set

$$\mathbf{ne}(\alpha) = \{\beta \in NE(\hat{X}) / \pi_*\beta = \alpha\}$$

where $NE(\hat{X})$ is the cone of effective 1-cycles in \hat{X} .

THEOREM 0.5. — *The irreducible components of $\mathbf{Hom}_\alpha(\mathbb{P}^1, X)$ of the scheme of morphisms from \mathbb{P}^1 to X of class α are indexed by $\mathbf{ne}(\alpha)$.*

The same kind of results are true for other special Schubert varieties (for example for cominuscule Schubert varieties, this will be studied in [Pe2]). It is also the case for cones over homogeneous varieties (see [Pe5]).

Let us give an overview of the article. In paragraphs 1 and 2 we recall some basic notations, definitions and results on elements of the Weyl group and minuscule Schubert varieties. In the

third paragraph, we define the quiver associated to a reduced writing \tilde{w} and the corresponding configuration variety. We study basic properties of these varieties (Weil, Cartier and canonical divisors, 1-cycles and intersection formulae), and link them with the geometry of the quiver. In the fourth paragraph, we describe the particular geometry of a quiver associated to a minuscule Schubert variety and study the link with the geometry of the Schubert variety (Weil, Cartier and canonical divisors, 1-cycles and intersection formulae). In the fifth paragraph, we construct and study a generalisation of Bott-Samelson resolution which is between the Schubert variety and the Bott-Samelson resolution. In the sixth paragraph, we describe the geometry of this generalisation (one more time Weil, Cartier and canonical divisors, 1-cycles and intersection formulae) and describe all the relative Mori theory for a minuscule Schubert variety. In the last paragraph, we prove that all relative minimal models are IH -small and describe all IH -small resolutions of minuscule Schubert varieties. In this paragraph, in contrast with the rest of the paper, we use a case by case analysis. A general proof could be possible but we think it would be more complicated and would lead to much more combinatorics. In an appendix, we describe the quivers of minuscule Schubert variety. We use this description intensively in the last paragraph.

Contents

1	Notations	6
2	Minuscule Schubert varieties	6
3	Quivers and configuration varieties	7
3.1	Quiver associated to a reduced writing	7
3.2	Configuration varieties and Bott-Samelson resolution	7
3.3	Cycles on the configuration variety	8
3.3.1	A basis of the Chow ring	8
3.3.2	Ample divisors	9
4	Geometry of the quiver	11
4.1	Minuscule conditions on the quivers	11
4.2	Combinatoric decription of the minuscule quivers	12
4.3	Weil and Cartier divisors	15
4.4	Canonical divisor	15
5	Generalisation of Bott-Samelson's construction	18
5.1	Elementary construction	18
5.2	Construction of the resolution	18
5.3	Link with the Bott-Samelson resolution	20
5.4	Constructing generalised reduced decomposition	22
6	Relative Mori theory of minuscule Schubert varieties	26
6.1	Ample divisors and effective curves	26
6.2	Canonical divisor of $\widehat{X}(\widehat{w})$	28
6.3	Types of singularities	28
6.4	Description of the relative minimal and canonical models	29
7	Small IH-resolutions of minuscule Schubert varities	31
7.1	Necessary condition	32
7.2	Fibers	32
7.3	The case of A_n and D_n	34
7.3.1	The A_n case	35
7.3.2	The D_n case	39
7.4	Exceptional cases	45
7.4.1	Quadrics	45
7.4.2	The E_6 case	46
7.4.3	The E_7 case	47
7.5	Small resolution	50
7.6	Stringy polynomials	52
8	Appendix	55
8.1	Quivers of minuscule homogeneous varieties	55
8.2	Quivers of minuscule Schubert varieties	58

1 Notations

Let G be a semi-simple algebraic group, fix T a maximal torus and B a Borel subgroup containing T . Let us denote by Δ the set of all roots, by Δ^+ (resp. Δ^-) the set of positive (resp. negative) roots, by S the set of simple roots associated to the data (G, T, B) and by W the associated Weyl group. If P is a parabolic subgroup containing B we note W_P the subgroup of W corresponding to P . We will also denote by $\Sigma(P)$ the set of simple roots β such that $U_{-\beta} \not\subset P$.

DEFINITION 1.1. — *Let $w \in W$, let us denote by P^w the largest parabolic subgroup of G such that the morphism $\overline{BwB}/B \rightarrow \overline{BwP^w}/P^w$ is a P^w/B fibration.*

Let us also denote by P_w the stabiliser of $X(w) = \overline{BwP^w}/P^w$ in G/P^w .

DEFINITION 1.2. — *(i) Let $w \in W$, we define the support of w denoted by $\text{Supp}(w)$ to be the set of simple roots β such that s_β appears in a reduced writing of w . This set is independent of the reduced writing and only depends on w .*

(ii) We will denote by G_w the smallest reductive subgroup of G containing all the groups U_β for $\beta \in \text{Supp}(w)$. It is easy to see that there is an isomorphism

$$X(w) = \overline{P_w w P^w} / P^w \simeq \overline{(P_w \cap G_w) w (P^w \cap G_w)} / (P^w \cap G_w).$$

(iii) Let us also define the boundary of G_w denoted by $\partial(G_w)$ to be the set of simple roots β not contained in $\text{Supp}(w)$ and non commuting with w .

2 Minuscule Schubert varieties

In this paragraph we recall the notion of minuscule weight and study the related homogeneous and Schubert varieties. Our basic reference will be [LMS].

DEFINITION 2.1. — *Let ϖ be a fundamental weight,*

(i) we say that ϖ is minuscule if we have $\langle \alpha^\vee, \varpi \rangle \leq 1$ for all positive root $\alpha \in \Delta^+$;

(ii) we say that ϖ is cominuscule if $\langle \alpha_0^\vee, \varpi \rangle = 1$ where α_0 is the longest root.

With the notation of N. Bourbaki [Bou], the minuscule and cominuscule weights are:

Type	minuscule	cominuscule
A_n	$\varpi_1 \cdots \varpi_n$	same weights
B_n	ϖ_n	ϖ_1
C_n	ϖ_1	ϖ_n
D_n	ϖ_1, ϖ_{n-1} and ϖ_n	same weights
E_6	ϖ_1 and ϖ_6	same weights
E_7	ϖ_7	same weight
E_8	none	none
F_4	none	none
G_2	none	none

DEFINITION 2.2. — Let ϖ be a minuscule weight and let P_ϖ be the associated parabolic subgroup. The homogeneous variety G/P_ϖ is then said to be minuscule. The Schubert varieties of a minuscule homogeneous variety are called minuscule Schubert varieties.

An element $w \in W$ is said to be minuscule if G/P^w is a minuscule homogeneous variety.

REMARK 2.3. — To study minuscule homogeneous varieties and their Schubert varieties, it is sufficient to restrict ourselves to simply-laced groups.

In fact the variety G/P_{ϖ_n} with $G = \text{Spin}_{2n+1}$ is isomorphic to the variety $G'/P'_{\varpi_{n+1}}$ with $G' = \text{Spin}_{2n+2}$ and there is a one to one correspondence between Schubert varieties thanks to this isomorphism. The same situation occurs with G/P_{ϖ_1} , $G = \text{Sp}_{2n}$ and G'/P'_{ϖ_1} , $G' = \text{SL}_{2n}$.

3 Quivers and configuration varieties

3.1 Quiver associated to a reduced writing

Let P_ϖ be a parabolic subgroup of G associated to a minuscule weight ϖ , let us consider an element $\bar{w} \in W/W_{P_\varpi}$ and let w be the shortest element in the class \bar{w} . To any reduced writing

$$w = s_{\beta_1} \cdots s_{\beta_r} \tag{1}$$

of w in terms of simple reflections (for all $i \in [1, r]$, we have $\beta_i \in S$) we associate a quiver with colored vertices. Let us first give the following

DEFINITION 3.1. — For a fixed reduced writing (1) of w , we define the successor $s(i)$ (resp. the predecessor $p(i)$) of an element $i \in [1, r]$ by $s(i) = \min\{j \in [1, r] \mid j > i \text{ and } \beta_j = \beta_i\}$ resp. by $p(i) = \max\{j \in [1, r] \mid j < i \text{ and } \beta_j = \beta_i\}$.

Now we can define the quiver $Q_{\tilde{w}}$ associated to the reduced writing (1) of w (we will denote by \tilde{w} the data of w with such a reduced writing):

DEFINITION 3.2. — Let us denote by $Q_{\tilde{w}}$ the quiver whose set of vertices is in bijection with the set $[1, r]$ and whose arrows are given in the following way: there is an arrow from i to j if $\langle \beta_j^\vee, \beta_i \rangle \neq 0$ and $i < j < s(i)$.

This quiver comes with a coloration of its edges by simple roots thanks to the application $\beta : [1, r] \rightarrow S$ such that $\beta(i) = \beta_i$.

REMARK 3.3. — It is equivalent to give the reduced writing \tilde{w} or the quiver $Q_{\tilde{w}}$. This quiver seems to be the same as the one defined by S. Zelikson [Zel] for ADE types.

3.2 Configuration varieties and Bott-Samelson resolution

In this paragraph, we associate to any quiver $Q_{\tilde{w}}$, a configuration variety $\tilde{X}(\tilde{w})$ (see also [Ma]). Let x be an element in G/B and β_i a simple root. Let us denote by B_{β_i} the parabolic subgroup generated by B and $U_{-\beta_i}$. We have a projection morphism $\pi_{\beta_i} : G/B \rightarrow G/B_{\beta_i}$ whose fibers are isomorphic to \mathbb{P}^1 and we denote by $\mathbb{P}(x, \beta_i)$ the projective line $\pi_{\beta_i}^{-1}(\pi_{\beta_i}(x))$.

DEFINITION 3.4. — Let $Q_{\tilde{w}}$ a quiver associated to a writing \tilde{w} of w . We define the configuration variety $\tilde{X}(\tilde{w})$ by

$$\tilde{X}(\tilde{w}) = \left\{ (x_1, \dots, x_r) \in \prod_{i=1}^r G/B \mid x_0 = 1 \text{ and } x_i \in \mathbb{P}(x_{i-1}, \beta_i) \text{ for all } i \in [1, r] \right\}.$$

REMARK 3.5. — (i) If we denote by P_{β_i} the maximal parabolic not containing $U_{-\beta_i}$, the restriction of the morphism $G/B \rightarrow G/P_{\beta_i}$ to $\mathbb{P}(x, \beta_i)$ is an isomorphism so that $\tilde{X}(\tilde{w})$ is isomorphic to

$$\left\{ (x_1, \dots, x_r) \in \prod_{i=1}^r G/P_{\beta_i} \mid x_0 = 1 \text{ and } x_i \in \mathbb{P}(x_{i-1}, \beta_i) \text{ for all } i \in [1, r] \right\}.$$

(ii) In his article [Ma], P. Magyar shows that the variety $\tilde{X}(\tilde{w})$ is isomorphic to the classical Bott-Samelson variety described for example in [De].

Because the element w is the shortest in the class \bar{w} and because the writing \tilde{w} is reduced, the last root β_r has to be the only simple root β such that $\langle \varpi^\vee, \beta \rangle = 1$ and $P_r = P_\varpi$. The image of the projection morphism $\pi : \tilde{X}(\tilde{w}) \rightarrow G/P_r$ is the minuscule Schubert variety $X(\bar{w})$. The morphism $\pi : \tilde{X}(\tilde{w}) \rightarrow X(\bar{w})$ is birational.

3.3 Cycles on the configuration variety

3.3.1 A basis of the Chow ring

In this paragraph, we describe some particular elements in the Chow ring $A_*(\tilde{X}(\tilde{w}))$. We describe a basis of this ring as the cones of ample divisors and effective curves. We calculate the canonical divisor in terms of these basis.

For all $k \in [1, r]$, let us denote by X_k the image of $\tilde{X}(\tilde{w})$ in the product $\prod_{i=1}^k G/P_{\beta_i}$ and $X_0 = \{x_0\}$. We have natural projection morphism $f_k : X_k \rightarrow X_{k-1}$ for all $k \in [1, r]$ which are \mathbb{P}^1 -fibration. The morphism $\sigma_k : X_{k-1} \rightarrow X_k$ defined by $\sigma_{k-1}(x_1, \dots, x_{k-1}) = (x_1, \dots, x_{k-1}, x_{p(k)})$ with $x_{p(k)} = 1$ if $p(k)$ does not exist is a section of f_k . We recover in this way the structure of $\tilde{X}(\tilde{w})$ as a tower of \mathbb{P}^1 -fibrations with sections described in [De].

Let us define the divisors $Z_i = f_r^{-1} \cdots f_{i+1}^{-1} \sigma_i(X_{i-1})$. The divisors $(Z_i)_{i \in [1, r]}$ have normal crossing (cf. for example [De]). We have

$$Z_i = \left\{ (x_1, \dots, x_r) \in \tilde{X}(\tilde{w}) \mid x_i = x_{p(i)} \right\}$$

in the configuration variety with $x_{p(i)} = 1$ if $p(i)$ does not exist. Then for any subset K of $[1, r]$, one defines $Z_K = \bigcap_{i \in K} Z_i$. Let us recall the following (see for example [De]):

FACT 3.6. — The image by π of Z_K is the Schubert subvariety $X(y)$ where y is the longest element that can be written as a subword of \tilde{w} without the terms s_i for $i \in K$.

Denote by ξ_i the class of Z_i in $A^*(\tilde{X}(\tilde{w}))$. These classes form a basis of the Chow ring.

THEOREM 3.7. — (Demazure [De] Par. 4. prop. 1) *The Chow ring $A^*(\tilde{X}(\tilde{w}))$ of $\tilde{X}(\tilde{w})$ is isomorphic over \mathbb{Z} to*

$$\frac{\mathbb{Z}[\xi_1, \dots, \xi_r]}{\left(\xi_i \cdot \sum_{j=1}^i \langle \alpha_j^\vee, \alpha_i \rangle \xi_j \text{ for all } i \in [1, r] \right)}.$$

Let us recall the following result (which is no longer true if the writing is not reduced):

PROPOSITION 3.8. — [LT] *The divisors $(\xi_i)_{1 \leq i \leq r}$ form a basis of the cone of effective divisors.*

Let us denote by T_i the pull-back on $\tilde{X}(\tilde{w})$ of the relative tangent sheaf of the fibration f_i . Define the classical sequence of roots $(\alpha_i)_{i \in [1, r]}$ associated to \tilde{w} by $\alpha_1 = \beta_1$, $\alpha_2 = s_{\beta_1}(\beta_2)$, \dots , $\alpha_r = s_{\beta_1} \cdots s_{\beta_{r-1}}(\beta_r)$. We denote by C_i the curve Z_K with $K = [1, r] \setminus \{i\}$ and recall some formulae in the ring $A^*(\tilde{X}(\tilde{w}))$ given in [Pe1] corollary 3.8 and propositions 3.3 and 3.11:

PROPOSITION 3.9. — *We have the following formulae:*

$$[C_i] \cdot \xi_j = \begin{cases} 0 & \text{for } i > j \\ 1 & \text{for } i = j \\ \langle \beta_i^\vee, \beta_j \rangle & \text{for } i < j \end{cases}, \quad [C_i] \cdot T_j = \begin{cases} 0 & \text{for } i > j \\ \langle \beta_i^\vee, \beta_j \rangle & \text{for } i \leq j \end{cases} \quad \text{and} \quad T_i = \sum_{k=1}^i \langle \alpha_k^\vee, \alpha_i \rangle \cdot \xi_k.$$

This proposition and the formula $c_1(\tilde{X}(\tilde{w})) = \sum_{i=1}^r T_i$ gives:

$$c_1(\tilde{X}(w)) = -K_{\tilde{X}(w)} = \sum_{k=1}^r \left(\sum_{i=k}^r \langle \alpha_k^\vee, \alpha_i \rangle \right) \xi_k.$$

3.3.2 Ample divisors

In this paragraph, we describe the ample divisors on $\tilde{X}(\tilde{w})$. This has already been done in [LT] but we rephrase it in terms of configuration varieties. We also get a description of the Mori cone (see [Mat] for references on this cone).

We have natural morphisms $p_i : \tilde{X}(\tilde{w}) \rightarrow G/P_{\beta_i}$ and as P_{β_i} is maximal, the Picard group of G/P_{β_i} is generated by a very ample divisor $\mathcal{O}_{G/P_{\beta_i}}(1)$ and we define on $\tilde{X}(\tilde{w})$ the invertible sheaf $\mathcal{L}_i = p_i^*(\mathcal{O}_{G/P_{\beta_i}}(1))$. These sheaves will form a basis of the ample cone.

We also define particular curves Y_i for $i \in [1, r]$ on $\tilde{X}(\tilde{w})$ by:

$$Y_i = \left\{ (x_1, \dots, x_r) \in \tilde{X}(\tilde{w}) \mid x_j = 1 \text{ for } j \neq i \right\}.$$

The following lemma shows that Y_i is a curve isomorphic to $\mathbb{P}(1, \beta_i)$.

LEMMA 3.10. — *For any $x_i \in \mathbb{P}(\bar{1}, \beta_i)$, the element $(x_j)_{j \in [1, r]}$ of $\prod_{j=1}^r G/P_{\beta_j}$ such that $x_j = \bar{1}$ for all $j \neq i$ is in the configuration variety $\tilde{X}(\tilde{w})$.*

PROOF — We only have to prove that for any x_i in $\mathbb{P}(\bar{1}, \beta_i) = B_{\beta_i}/B$, we have $\bar{1} \in \mathbb{P}(x_i, \beta_{i+1})$.

The element x_i can be lifted in $b_i \in B_{\beta_i}$. The elements of $\mathbb{P}(x_i, \beta_{i+1})$ are the classes of elements of the form $b_i b_{i+1} \in B_{\beta_i} B_{\beta_{i+1}}$. If $\beta_{i+1} \neq \beta_i$ (this is always the case if the writing is reduced), then $B_{\beta_i} \subset P_{\beta_{i+1}}$. In this case we set $b_{i+1} = 1$ so that the class of $b_i b_{i+1} \in \mathbb{P}(x_i, \beta_{i+1})$ is $\bar{1}$ in $G/P_{\beta_{i+1}}$. If $\beta_{i+1} = \beta_i$, then we set $b_{i+1} = b_i^{-1}$ to get the result. \square

The definitions of the curves Y_i and the line bundles L_i yield to following:

PROPOSITION 3.11. — *We have the formula $L_i \cdot [Y_j] = \delta_{i,j}$. In other words the families $(L_i)_{i \in [1,r]}$ and $([Y_i])_{i \in [1,r]}$ are dual to each other.*

Let us prove that the family $([Y_i])_{i \in [1,r]}$ forms a basis of $A_1(\tilde{X}(\tilde{w}))$.

PROPOSITION 3.12. — *For all $i \in [1,r]$, we have $[Y_i] = [C_i] - [C_{s(i)}]$ (where $[C_{s(i)}] = 0$ if $s(i)$ doesn't exist).*

In consequence, the classes $([Y_i])_{i \in [1,r]}$ form a basis of $A_1(X_Q)$ over \mathbb{Z} and the classes $(L_i)_{i \in [1,r]}$ form a basis of $A^1(X_Q)$ over \mathbb{Z} .

PROOF — On the first hand, the curve C_i is given by the equations $x_j = x_{p(j)}$ for $j \neq i$. This means that for $j < i$, we have $x_j = 1$ and for all j with $\beta_j \neq \beta_i$, we also have $x_j = 1$. The only indices k for which x_k may be different from 1 are such that $k = s^n(i)$ for some $n \in \mathbb{N}$. For such a k , we have the equality $x_k = x_i$. Denote by $n(i)$ the biggest integer n such that $s^n(i)$ exists. The curve C_i (resp. $C_{s(i)}$) is the diagonal in the product

$$\prod_{k=0}^{n(i)} \mathbb{P}(1, \beta_{s^k(i)}) \quad \text{resp.} \quad \prod_{k=1}^{n(i)} \mathbb{P}(1, \beta_{s^k(i)}).$$

On the other hand, the curve Y_i corresponds to the first factor of the first product. In this product we thus have the required equality. \square

We can now describe the ample cone (see also [LT]) and the cone of effective curves.

COROLLARY 3.13. — *The cone of ample divisors is generated by the classes L_i and the cone of effective curves is generated by the classes $[Y_i]$. All ample divisors are very ample.*

PROOF — Let D be ample on $\tilde{X}(\tilde{w})$, then $a_i = D \cdot [Y_i]$ is a positive integer. Because of proposition 3.11, we have $D = \sum_{i=1}^r a_i L_i$ and D lies in the cone generated by the L_i .

Conversely, any divisor $\sum_{i=1}^r a_i L_i$ with $a_i > 0$ gives the embedding of $\tilde{X}(\tilde{w})$ obtained by composing the inclusion in the product $\prod_{i=1}^r G/P_{\beta_i}$ with the Veronese morphism given by the very ample sheaf $\bigotimes_{i=1}^r \mathcal{O}_{G/P_{\beta_i}}(a_i)$.

In the same way we get the result on effective curve. \square

Finally we calculate the divisors L_i in terms of the basis $(\xi_k)_{k \in [1,r]}$:

PROPOSITION 3.14. — *The k^{th} coordinate of L_i in the basis $(\xi_i)_{i \in [1, r]}$ is 0 if $k > i$, 1 if $k = i$ and is given by the following formulae if $k < i$ and $\beta_k = \beta_i$ (resp. $\beta_k \neq \beta_i$):*

$$1 + \sum_{j=k+1, \beta_j=\beta_i}^i \langle \alpha_k^\vee, \alpha_j \rangle \left(\text{resp.} \sum_{j=k+1, \beta_j=\beta_i}^i \langle \alpha_k^\vee, \alpha_j \rangle \right).$$

In particular we have the following simple formula

$$L_r = \sum_{k=1}^r \xi_k.$$

PROOF — Let us recall from [Pe1] lemma 4.5 that the following classes of curves $[\widehat{C}_i] = [C_i] + \sum_{k=i+1}^n \langle \alpha_i^\vee, \alpha_k \rangle [C_k]$ form a dual basis to $(\xi_i)_{i \in [1, r]}$. The k^{th} coordinate is thus given by the intersection $L_i \cdot [\widehat{C}_k]$.

For this we will need the formula coming directly from propositions 3.11 and 3.12

$$L_i \cdot [C_j] = \begin{cases} 1 & \text{for } i > j \text{ and } \beta_i = \beta_j \\ 0 & \text{otherwise.} \end{cases}$$

Applying this gives the first formula. For the case of L_r , the formula is a consequence of the following formula from [Pe1] corollary 2.18:

$$\sum_{j=k+1, \beta_j=\beta_r}^r \langle \alpha_k^\vee, \alpha_j \rangle = \begin{cases} 1 & \text{if } \beta_k \neq \beta_r \\ 0 & \text{if } \beta_k = \beta_r. \end{cases}$$

□

4 Geometry of the quiver

In this paragraph, we give an explicit description of the quiver $Q_{\bar{w}}$ given by the reduced writing $w = s_{\beta_1} \cdots s_{\beta_r}$ of the shortest element in the class $\bar{w} \in W/W_{P_\varpi}$. We also define some invariants of the quiver and deduce some consequences on the geometry of the Schubert variety.

4.1 Minuscule conditions on the quivers

Set $w_i = s_{\beta_i} \cdots s_{\beta_r}$ for $i \in [1, r]$ and $w_{r+1} = 1$ and let us first recall the following result from [LMS] (proof of theorem 3.1):

FACT 4.1. — *We have $\langle \beta_{i-1}^\vee, w_i(-\varpi) \rangle = -1$ for all $i \in [2, r+1]$. In consequence we have for all $i \in [2, r]$:*

$$w_i(-\varpi) = -\varpi + \beta_r + \cdots + \beta_i.$$

The following proposition describes all possible quivers for minuscule Schubert varieties.

PROPOSITION 4.2. — *Geometry of the quiver.*

- (i) There is no arrow from the vertex r and β_r is the unique simple root with $\langle \beta_r^\vee, \varpi \rangle = 1$.
- (ii) If a vertex $i < r$ of the quiver is such that $s(i)$ does not exist, then there is a unique arrow from i . If k is the end of the arrow we have $\langle \beta_i^\vee, \beta_k \rangle = -1$.
- (iii) If a vertex i of the quiver is such that $s(i)$ exists, then there are exactly two arrows from i . If k_1 and k_2 are the end of these arrows we have $\langle \beta_i^\vee, \beta_{k_1} \rangle = \langle \beta_i^\vee, \beta_{k_2} \rangle = -1$.

PROOF — (i) The previous fact shows that we have $\langle \beta_r^\vee, \varpi \rangle = 1$.

(ii) Let i be such a vertex. In particular we have $\beta_i \neq \beta_r$ and $\langle \beta_i^\vee, \varpi \rangle = 0$. The previous fact gives $\langle \beta_i^\vee, -\varpi + \beta_r + \dots + \beta_i \rangle = \langle \beta_i^\vee, w_{i+1}(-\varpi) \rangle = -1$ and thus

$$\sum_{k=i+1}^r \langle \beta_i^\vee, \beta_k \rangle = -1.$$

We conclude because every term of this sum have to be either 0 or -1 .

(iii) Let i be such a vertex. The same calculation as above shows that $\sum_{k=i+1}^r \langle \beta_i^\vee, \beta_k \rangle = \sum_{k=s(i)+1}^r \langle \beta_i^\vee, \beta_k \rangle = -1$ if $\beta_i \neq \beta_r$ and $\sum_{k=i+1}^r \langle \beta_i^\vee, \beta_k \rangle = \sum_{k=s(i)+1}^r \langle \beta_i^\vee, \beta_k \rangle = 0$ if $\beta_i = \beta_r$. In particular we always have

$$\sum_{k=i+1}^{s(i)} \langle \beta_i^\vee, \beta_k \rangle = 0.$$

We conclude because the only positive term is $\langle \beta_i^\vee, \beta_{s(i)} \rangle = 2$ and every other term of this sum have to be either 0 or -1 . \square

REMARK 4.3. — (i) One can deduce from this result (see [Pe2]) the fact already proved by J. R. Stembridge [St] that the reduced writing \tilde{w} is unique modulo commutation relations (there is no braid relation). We can thus write Q_w instead of $Q_{\tilde{w}}$ and call it the quiver associated to the minuscule Schubert variety $X(w)$.

Furthermore, because of this unicity and the fact that between i and $s(i)$ there are always two vertices, the writing deduced from a quiver satisfying the conditions of the preceding proposition is always reduced and the quivers satisfying the conditions are always quivers associated to a minuscule Schubert variety.

(ii) A minuscule quiver is always connected: there is a path from any vertex i to the last vertex r .

4.2 Combinatoric description of the minuscule quivers

It is easy from proposition 4.2 to describe the quivers Q_ϖ of a minuscule homogeneous variety G/P_ϖ (see appendix for a list of these quivers). We now describe the quivers of minuscule Schubert varieties in G/P_ϖ as subquivers of Q_ϖ . Define a natural partial order on the quiver:

DEFINITION 4.4. — (i) We denote by \preceq the partial order on the vertices of the quiver generated by the relations $i \preceq j$ if there exists an arrow from i to j .

(ii) Let A be a totally unordered set of vertices of the quiver Q_ϖ for the partial order \preceq . We denote by $Q_{\bar{A}}$ the full subquiver of Q_ϖ with vertices $i \in Q_\varpi$ such that there exists $a \in A$ with $i \preceq a$ and by Q_A the full subquiver of Q_ϖ whose vertices are not vertices of $Q_{\bar{A}}$.

PROPOSITION 4.5. — *The quiver of Schubert varieties in G/P_ϖ are in one to one correspondence with the subquivers Q_A of Q_ϖ for A any totally unordered set of vertices of Q_ϖ .*

With this correspondence, the Bruhat order is given by the inclusion of quivers.

PROOF — Let $X(w) \subset G/P_\varpi$ a Schubert variety and denote by $w_0 \in W$ the only element such that $X(w_0) = G/P_\varpi$. There exists a sequence $(\beta_1, \dots, \beta_i)$ of simple roots such that $w_0 = s_{\beta_1} \cdots s_{\beta_i} w$. Taking a reduced writing $w = s_{\beta_{i+1}} \cdots s_{\beta_r}$ of w we get a reduced writing of w_0 which is unique modulo commutation relations.

The vertices of the quiver Q_ϖ are indexed by $[1, r]$. Denote by $A = \{i_1, \dots, i_k\}$ the set of maximal elements for the partial order \preceq of the set $[1, i]$. The set A is totally unordered and the quiver associated to $X(w)$ is Q_A .

The fact that this is a one to one correspondence comes from the unicity of the reduced writing. \square

REMARK 4.6. — *As a corollary one can prove (see [Pe2]) the following classical result on the Bruhat order (see for example [LMS]): the Bruhat order in W/W_P is generated by simple reflexions. In other words, all Schubert divisors are mobile.*

Finally let us define some particular vertices of these quivers. In the following definition Q_w is the quiver of a minuscule Schubert variety $X(w)$.

DEFINITION 4.7. — (i) We call pic any vertex of Q_w minimal for the partial order \preceq . We denote by $p(Q_w)$ the set of pics of Q_w .

(ii) We call hole of the quiver Q_w any vertex i of Q_w such that $p(i)$ does not exist and there are exactly two vertices $j_1 \preceq i$ and $j_2 \preceq i$ in Q_w with $\langle \beta_i^\vee, \beta_{j_k} \rangle \neq 0$ for $k = 1, 2$.

We will also call a hole of Q_w any $i \in Q_\varpi \setminus Q_w$ such that $s(i)$ does not exist in Q_ϖ and $\beta_i \in \partial(G_w)$. Such a hole will be called a virtual hole. We denote by $t(Q_w)$ the set of holes of Q_w .

(iii) The height $h(i)$ of a vertex i is the largest n such that there exists a sequence $(i_k)_{k \in [1, n]}$ of vertices with $i_1 = 1$, $i_n = r$ and such that there is an arrow from i_k to i_{k+1} for all $k \in [1, n-1]$.

REMARK 4.8. — (i) If $Q_w = Q_A$ as in definition 4.4 then $t(Q_w) = A$.

(ii) The height is well defined because there is at least one path from any vertex i to the last vertex r .

The following proposition gives a recursive way to calculate the height of a vertex.

PROPOSITION 4.9. — *Let Q be a quiver associated to a minuscule Schubert variety and i a vertex of this quiver. We have the following cases:*

- if $s(i)$ does not exist, then there exists a unique $k \succ i$ with $\langle \beta_k^\vee, \beta_i \rangle = -1$ and we have

$$h(i) = h(k) + 1.$$

- If $s(i)$ exists, then there exists a non negative integer n and a sequence $(j_k, j'_k)_{k \in [0, n+1]}$ of vertices with $j_0 = i$, $j'_k = s(j_k)$ for $k \in [0, n]$, $\beta_{j_{n+1}} \neq \beta_{j'_{n+1}}$, $\langle \beta_{j_k}, \beta_{j_{k+1}} \rangle = -1$ and $\langle \beta'_{j_k}, \beta'_{j_{k+1}} \rangle = -1$ for $k \in [0, n]$ and $j_0 \preccurlyeq \dots \preccurlyeq j_n \preccurlyeq j_{n+1}, j'_{n+1} \preccurlyeq j'_n \preccurlyeq \dots \preccurlyeq j'_0$. In this case we have

$$h(j_k) = 2n + 2 - k + h(s(i)) \quad \text{and} \quad h(j'_k) = k + h(s(i)) \quad \text{for all } k \in [0, n+1].$$

PROOF — We prove these formulae by descending induction on i . If $i = r$ then $h(i) = 1$. Assume the proposition is true for all $j > i$. In the first case, any sequence of arrows from i to r has to pass through the vertex k and we have $h(i) = h(k) + 1$.

The first to prove in the second case is the existence of the sequence $(j_k, j'_k)_{k \in [0, n+1]}$. Let us denote by j_1 and j'_1 the two vertices j such that there is an arrow from i to j . If $\beta_{j_1} \neq \beta_{j'_1}$ then set $n = 0$ and we are done. Otherwise, assume (for example) that $j_1 < j'_1$ then $j'_1 = s(j_1)$. Indeed, if it was not the case there would exist $k \in [j_1, j'_1]$ and in particular $k < s(i)$ with $\beta_k = \beta_{j_1}$ thus $\langle \beta_i^\vee, \beta_k \rangle = -1$. By construction of the quiver, there must be an arrow from i to k and thus at least three arrows from i . This is impossible by proposition 4.2. We can construct with (j_1, j'_1) a couple (j_2, j'_2) in the same way and by induction a sequence $(j_k, j'_k)_k$. As long as $\beta_{j_k} = \beta_{j'_k}$ we can go on. This has to stop because the quiver is finite.

The formula is now clear because any sequence from i to r has to go through j_1 or j'_1 . As the height of j_1 is bigger than the one of $j'_1 = s(j_1)$ by induction we must have $h(i) = h(j_1) + 1$ and we conclude one more time by induction. \square

REMARK 4.10. — By changing the order of commuting factors, we may assume in the preceding proposition that $k = i + 1$ in the first case and that $j_k = i + k$ and $j'_k = i + 2n + 3 - k$ for all $k \in [0, n + 1]$ in the second one.

We can now describe the stabiliser of a Schubert variety $X(w)$ thanks to its quiver Q_w .

PROPOSITION 4.11. — Let J be the set of simple roots not in $\beta(t(Q_w))$. The stabiliser of $X(w)$ is the parabolique subgroup P_J generated by B and the groups $U_{-\beta}$ with $\beta \in J$.

PROOF — A simple root β is such that $U_{-\beta} \subset P$ if and only if $s_\beta w < w$ (for the Bruhat order). But from unicity of reduced writing and our characterisation (proposition 4.2) of quivers associated to a reduced writing, we see that this implies that β has to be in $\beta(t(Q_w))$. \square

COROLLARY 4.12. — Let $Q_{w'}$ be the quiver of a Schubert subvariety $X(w')$ of $X(w)$ stable under $P_J = \text{Stab}(X(w))$. Then $\beta(t(Q_{w'})) \subset \beta(t(Q_w))$.

4.3 Weil and Cartier divisors

In this paragraph, we describe thanks to the quiver the Weil and Cartier divisors of a minuscule Schubert variety $X(w)$. We also compute the canonical sheaf of $X(w)$.

PROPOSITION 4.13. — *The group $\text{Weil}(X(w))$ of Weil-divisors is the free \mathbb{Z} -module generated by the classes $D_i := \pi_* \xi_i$ for $i \in p(Q_w)$*

The Picard group $\text{Pic}(X(w)) \subset \text{Weil}(X(w))$ is isomorphic to \mathbb{Z} and is generated by the element $L(w) := \pi_ L_r = \mathcal{O}_{G/P_{\beta_r}}(1)|_{X(w)}$. We have the formula*

$$L(w) = \sum_{i \in p(Q_w)} D_i.$$

PROOF — It is well known (see for example [Br]) that the Picard group is isomorphic to \mathbb{Z} and generated by $L(w)$ and that the group of Weil divisors is free generated by the divisorial Schubert varieties. These varieties are the images by $\pi : \tilde{X}(\tilde{w}) \rightarrow X(w)$ of the non contracted divisors Z_i . Now the divisor Z_i is the configuration variety obtained from the quiver Q with the vertex i removed (the arrows are reorganised as in definition 3.1). According to the fact 3.6, the image of Z_i is not contracted if and only if the quiver is reduced (ie correspond du a reduced writing). It is clear that this can only be the case for $i \in p(Q_w)$.

The last formula is an application of corollary 3.14. We recover the well known fact that all Schubert divisors are of multiplicity one in the minuscule case. \square

COROLLARY 4.14. — *A minuscule Schubert variety is locally factorial if and only if its quiver has a unique pic.*

4.4 Canonical divisor

The Schubert varieties are singular and in general not Gorenstein (see [WY] for a characterisation of Gorenstein Schubert varieties for GL_n). We can therefore not define the canonical divisor as a Cartier divisor.

The canonical divisor $K_{X(w)}$ of a Schubert variety $X(w)$ is well defined as a Weil divisor thanks to the divisor of a 1-form on $X(w)$. The properties of Schubert varieties (they are normal, Cohen-Macaulay with rational singularities) and the Bott-Samelson resolution $\pi : \tilde{X}(\tilde{w}) \rightarrow X(w)$ enables however to calculate $K_{X(w)}$ by $K_{X(w)} = \pi_*(K_{\tilde{X}(\tilde{w})})$ (see for example [BK] paragraph 3.4).

Let us denote by $h(w)$ the lowest height of a pic in Q_w (the quiver associated to $X(w)$), we have the following:

PROPOSITION 4.15. — *We have the formula*

$$-K_{X(w)} = \sum_{i \in p(Q_w)} (h(i) + 1) D_i = (h(w) + 1) L(w) + \sum_{i \in p(Q_w)} (h(i) - h(w)) D_i.$$

PROOF — The second part of the formula comes from the first one and proposition 4.13.

To prove the first part, we use the fact that $K_{X(w)} = \pi_*(K_{\tilde{X}(\tilde{w})})$ and the formula of paragraph 3.3.1. We are left to prove the following:

LEMMA 4.16. — *We have the formula:*

$$\sum_{k=i}^r \langle \alpha_i^\vee, \alpha_k \rangle = \sum_{k \succcurlyeq i} \langle \alpha_i^\vee, \alpha_k \rangle = h(i) + 1$$

PROOF — We proceed by descending induction and use the proposition 4.9 and the remark 4.10. We have the following two cases:

- if $s(i)$ does not exist, then with $\langle \beta_{i+1}^\vee, \beta_i \rangle = -1$.
- If $s(i)$ exists, then there exists a non negative integer n and a sequence $(j_k, j'_k)_{k \in [0, n+1]}$ of vertices with $j_0 = i$, $j'_k = s(j_k)$ for $k \in [0, n]$, $\beta_{j_{n+1}} \neq \beta_{j'_{n+1}}$, $\langle \beta_{j_k}, \beta_{j_{k+1}} \rangle = -1$ and $\langle \beta'_{j_k}, \beta'_{j_{k+1}} \rangle = -1$ for $k \in [0, n]$ and $j_0 \preccurlyeq \dots \preccurlyeq j_n \preccurlyeq j_{n+1}, j'_{n+1} \preccurlyeq j'_n \preccurlyeq \dots \preccurlyeq j'_0$. Furthermore we may assume that $j_k = i + k$ and $j'_k = i + 2n + 3 - k$ for all $k \in [0, n + 1]$.

We proceed by descending induction on i . If $i = r$, there is only one term and the sum is $\langle \alpha_r^\vee, \alpha_r \rangle = 2 = h(r) + 1$. We assume that the formula is true for all $j \geq i + 1$. Let us use the following sequence $\tilde{\alpha}_j = s_{\beta_r} \cdots s_{\beta_{r-j+2}}(\beta_{r-j+1})$ already introduced in [Pe1] and satisfying the equality $\langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r+1-i} \rangle = \langle \alpha_k^\vee, \alpha_i \rangle$.

Calculating $\tilde{\alpha}_{r+1-i} = s_{\beta_r} \cdots s_{\beta_{i+1}}(\beta_i)$ we find

$$\tilde{\alpha}_{r+1-i} = \begin{cases} \tilde{\alpha}_{r-i} + s_{\beta_r} \cdots s_{\beta_{i+2}}(\beta_i) & \text{in the first case;} \\ \tilde{\alpha}_{r-i-n} + \tilde{\alpha}_{r-i-n-1} - \tilde{\alpha}_{r-i-2n-2} & \text{in the second case;} \end{cases}$$

Let us now calculate the sum

$$\sum_{k \succcurlyeq i} \langle \alpha_k^\vee, \alpha_i \rangle = \sum_{k \succcurlyeq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r+1-i} \rangle.$$

In the first case, denote $\alpha = s_{\beta_r} \cdots s_{\beta_{i+2}}(\beta_i)$. If $j \geq i + 2$, then we have $\langle \beta_i^\vee, \beta_j \rangle = 0$ so that $\alpha = \beta_i$. Furthermore, the root $\tilde{\alpha}_{r+1-j}$ is a sum of simple roots contained in the set $\{\beta_j, \dots, \beta_r\}$ so for $j \geq i + 2$, we have $\langle \tilde{\alpha}_{r+1-j}^\vee, \alpha \rangle = 0$. In this case the sum equals:

$$\begin{aligned} \sum_{k \succcurlyeq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r+1-i} \rangle &= \sum_{k \succcurlyeq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i} \rangle + \sum_{k \succcurlyeq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \alpha \rangle \\ &= \sum_{k \succcurlyeq i+1} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i} \rangle + \langle \tilde{\alpha}_{r+1-i}^\vee, \tilde{\alpha}_{r-i} \rangle + \langle \tilde{\alpha}_{r+1-i}^\vee, \alpha \rangle + \langle \tilde{\alpha}_{r-i}^\vee, \alpha \rangle \\ &= h(i+1) + 1 - \langle \beta_i^\vee, \beta_{i+1} \rangle + \langle (\beta_i + \beta_{i+1})^\vee, \beta_i \rangle + \langle \beta_{i+1}^\vee, \beta_i \rangle \\ &= h(i+1) + 1 + 1 + 1 - 1 = h(i+1) + 2. \end{aligned}$$

In the second case, it is an easy exercise to see that the roots $\{\beta_k\}_{k \in [i, i+n+2]}$ form a diagram of type D_{n+2} (with the notations of [Bou] the root β_{i+k} is the $(k+1)$ -th root of the diagram). We can then calculate

$$\tilde{\alpha}_{r+1-i-k} = \begin{cases} s_{\beta_r} \cdots s_{\beta_{i+2n+4}} \left(\sum_{j=0}^{2n+2-k} \beta_{i+j} \right) & \text{for all } k \in [0, n+1] \\ s_{\beta_r} \cdots s_{\beta_{i+2n+4}} \left(\sum_{j=0}^{2n+3-k} \beta_{i+j} \right) & \text{for all } k \in [n+2, 2n+3] \end{cases}$$

The sum is in this case:

$$\begin{aligned} \sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r+1-i} \rangle &= \sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-n} \rangle + \sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-n-1} \rangle - \sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-2n-2} \rangle \\ &= \sum_{k \geq i+n+1} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-n} \rangle + \sum_{k \geq i+n+2} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-n-1} \rangle - \sum_{k \geq i+2n+3} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-2n-2} \rangle \\ &\quad + \sum_{k=0}^n \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-n} \rangle + \sum_{k=0}^{n+1} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-n-1} \rangle - \sum_{k=0}^{2n+2} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-2n+2} \rangle. \end{aligned}$$

The description of $\tilde{\alpha}_{r+1-i-k}$ shows that all roots $\tilde{\alpha}_{r+1-i-k}$ for $k \in [0, n]$ (resp. $k \in [1, 2n+2]$) have intersection 1 with $\tilde{\alpha}_{r-i-n}$ and $\tilde{\alpha}_{r-i-n-1}$ (resp. $\tilde{\alpha}_{r-i-2n-2}$) and we have the intersections $\langle \tilde{\alpha}_{r-i-n}^\vee, \tilde{\alpha}_{r-i-n-1} \rangle = 0$ and $\langle \tilde{\alpha}_{r+1-i}^\vee, \tilde{\alpha}_{r-i-2n-2} \rangle = 0$. This gives us the following formulae:

$$\sum_{k=0}^n \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-n} \rangle = n+1; \quad \sum_{k=0}^{n+1} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-n-1} \rangle = n+1; \quad \sum_{k=0}^{2n+2} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i-2n+2} \rangle = 2n+2.$$

Using the induction hypothesis we get:

$$\begin{aligned} \sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r+1-i} \rangle &= h(i+n+1) + 1 + h(i+n+2) + 1 - h(i+2n+3) - 1 + n + 1 + n + 1 - 2n + 2 \\ &= h(i+n+1) + h(i+n+2) + 1 - h(i+2n+3). \end{aligned}$$

We conclude in both cases thanks to proposition 4.9. \square

This lemma completes the proof of the proposition. \square

COROLLARY 4.17. — *The Schubert variety $X(w)$ is Gorenstein if and only if all the pics of its quiver have the same height. In this case we have $K_{X(w)} = (h(w) + 1)L(w)$.*

PROOF — The variety $X(w)$ is Gorenstein if and only if its canonical divisor is Cartier. The preceding formula shows that this is equivalent to the fact that all the pics of its quiver have the same height. \square

REMARK 4.18. — *For $Gl(n)$, we recover a particular case of the result of A. Woo and A. Yong [WY] on Gorenstein Schubert varieties.*

5 Generalisation of Bott-Samelson's construction

In this paragraph, we are going to construct some varieties $\widehat{X}(\widehat{w})$ together with a birational morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$. These constructions generalise the IH -small resolutions constructed by A. Zelevinsky [Ze] and P. Sankaran and P. Vanchinathan [SV1].

Recall that the Bott-Samelson varieties can be seen as a tower of \mathbb{P}^1 -fibrations coming from a reduced writing \widetilde{w} of an element w . Many generalisations of this construction (for example in [Ze] and [SV1] but also in [Pe3] and even in the general construction of C. Contou-Carrère [Co]) are constructed as towers of locally trivial fibrations with fibers isomorphic to a fixed homogeneous variety thanks to a more general decomposition \widehat{w} of w as a product of elements in the Weyl group. For the varieties $\widehat{X}(\widehat{w})$ we make the same construction but we allow locally trivial fibrations with fiber a locally factorial or gorenstein Schubert variety.

5.1 Elementary construction

Let us explain the following elementary construction. As in [De], the construction of $\widehat{X}(\widehat{w})$ will simply be a successive application of this elementary construction. Let $u \in W$ and Y a variety with action of P_Y a parabolic subgroup of G containing B and assume that $P^u \cap G_u \subset P_Y$. We define

$$\widehat{Y}(u) = \overline{(P_u \cap G_u)u(P^u \cap G_u)} \times^{(P^u \cap G_u)} Y.$$

LEMMA 5.1. — (i) *The variety $\widehat{Y}(u)$ is a locally trivial fibration over $X(u)$ with fibers isomorphic to Y .*

(ii) *Define the parabolic $P_{\widehat{Y}(u)}$ by $\Sigma(P_{\widehat{Y}(u)}) = (\Sigma(P_Y) \cap \text{Supp}(u)^c) \cup \partial(G_u) \cup (\Sigma(P_u) \cap \text{Supp}(u))$. There is an action of $P_{\widehat{Y}(u)}$ on $\widehat{Y}(u)$.*

PROOF — (i) The first part of the proposition comes from the isomorphism between $X(u)$ and $\overline{(P_u \cap G_u)u(P^u \cap G_u)} / (P^u \cap G_u)$.

(ii) For the second part, let us remark that one can replace the groups $P_u \cap G_u$ and $P^u \cap G_u$ by bigger groups A and B such that the natural map $\overline{(P_u \cap G_u)u(P^u \cap G_u)} / (P^u \cap G_u) \rightarrow \overline{AuB} / B$ is an isomorphism and $B \subset P_Y$. For example, we take A and B such that $\Sigma(A) = (\Sigma(P_Y) \cap \text{Supp}(u)^c) \cup \partial(G_u) \cup (\Sigma(P_u) \cap \text{Supp}(u))$ and $\Sigma(B) = (\Sigma(P_Y) \cap \text{Supp}(u)^c) \cup (\Sigma(P^u) \cap \text{Supp}(u))$. We have the required isomorphism and $B \subset P_Y$ (simply because $\Sigma(P_Y) \subset \Sigma(B)$). Then $\widehat{Y}(u)$ is isomorphic to $\overline{AuB} \times^B Y$ and A acts on $\widehat{Y}(u)$. \square

5.2 Construction of the resolution

Let us give the following:

DEFINITION 5.2. — (i) *Let $w \in W$, a writing $w = w_1 \cdots w_n$ where for all $i \in [1, n]$, we have $w_i \in W$ is called a generalised decomposition and denoted by \widehat{w} . If moreover we have the equality $l(w) = \sum_{i=1}^n l(w_i)$ then we will say that the generalised decomposition is reduced.*

(u) Let us associate to any generalised decomposition a sequence of parabolic subgroups $(P_i)_{i \in [1, n]}$ defined by $P_n = P_{w_n}$ and

$$\Sigma(P_i) = (\Sigma(P_{i+1}) \cap \text{Supp}(w_i)^c) \cup \partial(G_{w_i}) \cup (\Sigma(P_{w_i}) \cap \text{Supp}(w_i)).$$

(uu) We will say that a generalised reduced decomposition is admissible if for all $i \in [1, n-1]$ we have $P^{w_i} \cap G_{w_i} \subset P_{i+1}$.

(v) We will say that such a generalised reduced decomposition is good if for all $i \in [1, n-1]$ we have $P^{w_i} \cap G_{w_i} \subset P_{w_{i+1} \cdots w_n}$ and $\partial(G_{w_i}) \subset \Sigma(P_{w_i \cdots w_n})$.

PROPOSITION 5.3. — Let \hat{w} an admissible reduced generalised decomposition of a minuscule element w of W . One can define by descending induction on n the varieties $\hat{X}_i(\hat{w})$ by $\hat{X}_n(\hat{w}) = X(w_n)$ and for $i < n$ by $\hat{X}_i(\hat{w}) = \hat{Y}(w_i)$ where $Y = \hat{X}_{i+1}(\hat{w})$. The group $P_{\hat{X}_i(\hat{w})}$ is the group P_i .

Furthermore, if the decomposition is good, then the group $P_{\hat{X}_i(\hat{w})}$ is the group $P_{w_i \cdots w_n}$ and in particular any good generalised reduced decomposition is admissible.

PROOF — We proceed by induction. The variety $\hat{X}_n(\hat{w})$ is well defined and we have $P_{\hat{X}_n(\hat{w})} = P_{w_n}$. Assume that $\hat{X}_{i+1}(\hat{w})$ is well defined and that $P_{\hat{X}_{i+1}(\hat{w})} = P_{i+1}$. To prove that $\hat{X}_i(\hat{w})$ exists, we have to prove that $P^{w_i} \cap G_{w_i} \subset P_{\hat{X}_{i+1}(\hat{w})}$ but it is the case by hypothesis. The fact that $P_{\hat{X}_i(\hat{w})} = P_i$ comes from lemma 5.1.

Now in the case of a good generalised reduced decomposition, we have to prove that $P_i = P_{\hat{X}_i(\hat{w})} = P_{w_i \cdots w_n}$. We know from lemma 5.1 that $\Sigma(P_{\hat{X}_i(\hat{w})}) = (\Sigma(\hat{X}_{i+1}(\hat{w})) \cap \text{Supp}(w_i)^c) \cup \partial(G_{w_i}) \cup (\Sigma(P_{w_i}) \cap \text{Supp}(w_i))$. Let $\beta \in \Sigma(P_{w_i \cdots w_n})$. If $\beta \in \text{Supp}(w_i)$ then β has to be a hole of the quiver of w_i so that $\beta \in \Sigma(P_{w_i})$ and $\beta \in \Sigma(P_{\hat{X}_i(\hat{w})})$. If $\beta \notin \text{Supp}(w_i)$ and $\beta \notin \partial(G_{w_i})$ then s_β commutes with w_i and we have $\beta \in \Sigma(P_{w_{i+1} \cdots w_n})$ and $\beta \in \Sigma(P_{\hat{X}_i(\hat{w})})$. Finally if $\beta \in \partial(G_{w_i})$ we also have $\beta \in \Sigma(P_{\hat{X}_i(\hat{w})})$.

Conversely, let $\beta \in \Sigma(P_{\hat{X}_i(\hat{w})})$. If $\beta \in \text{Supp}(w_i)$ then $\beta \in \Sigma(P_{w_i})$ so β corresponds to a hole of the quiver of w_i and thus have to be a hole of the quiver of $w_i \cdots w_n$. If $\beta \in \partial(G_{w_i})$ we are done by hypothesis and finally if β is neither in $\text{Supp}(w_i)$ nor in $\partial(G_{w_i})$ then $\beta \in \Sigma(P_{w_{i+1} \cdots w_n})$ by induction hypothesis. Thus β corresponds to a hole of the quiver of $w_{i+1} \cdots w_n$ and does not appear in the quiver of w_i . It is thus still a hole of the quiver of $w_i \cdots w_n$. \square

DEFINITION 5.4. — We denote by $\hat{X}(\hat{w})$ the variety $\hat{X}_1(\hat{w})$.

COROLLARY 5.5. — The variety $\hat{X}(\hat{w})$ is a tower of locally trivial fibrations f_i with fibers isomorphic to $X(w_i)$.

LEMMA 5.6. — Let \hat{w} an admissible reduced generalised decomposition of a minuscule element w of W . Let $i \in [1, n-1]$ and assume that for any couple $(\beta, \beta') \in \text{Supp}(w_i) \times \text{Supp}(w_{i+1})$ we have $\langle \beta^\vee, \beta' \rangle = 0$.

Then $w_i w_{i+1} = w_{i+1} w_i$, the generalised writing \hat{w}' given by $w = w'_1 \cdots w'_n$ where $w'_k = w_k$ for $k \notin \{i, i+1\}$, $w'_i = w_{i+1}$ and $w'_{i+1} = w_i$ is admissible and reduced and the morphisms $\hat{\pi} : \hat{X}(\hat{w}) \rightarrow X(w)$ and $\hat{\pi}' : \hat{X}(\hat{w}') \rightarrow X(w)$ are the same.

PROOF — We simply have to look at the following situation. Let A and B be parabolic subgroups of a group G and C and D be parabolic subgroups of a group G' . Assume that B and D act on a variety X and consider the variety $\overline{AuB} \times^B \overline{CvD} \times^D X$ (B acts on $\overline{CvD} \times^D X$ thanks to its action on X). It is isomorphic to $(\overline{AuB} \times \overline{CvD}) \times^{B \times D} X$ and the construction is completely symmetric.

Let us remark that the variety $\widehat{X}(\widehat{w})$ is also isomorphic to the variety $\widehat{X}(\widehat{w}'')$ where \widehat{w}'' is such that $w''_k = w_k$ for $k < i$, $w''_i = w_i w_{i+1}$ and $w''_k = w_{k+1}$ for $k > i + 1$. \square

Thanks to this lemma, we may assume that the support of any element w_i is connected (otherwise replace it by a product of elements having a connected support).

5.3 Link with the Bott-Samelson resolution

In this paragraph, we show that the Bott-Samelson resolution $\widetilde{X}(\widetilde{w})$ of a minuscule element factorises through any pseudo-resolution $\widehat{X}(\widehat{w})$ constructed above. And we view this variety as a projection from the Bott-Samelson resolution $\widetilde{X}(\widetilde{w})$.

Let \widehat{w} be an admissible generalised reduced writing of w and let us fix for any $i \in [1, n]$ a (unique) reduced writing $w_i = s_{1,i} \cdots s_{r_i,i}$ denoted \widehat{w}_i .

LEMMA 5.7. — *For any $i \in [1, n]$, the writing*

$$w = \left(\prod_{k=1}^i \prod_{j=1}^{r_k} s_{j,k} \right) \cdot \prod_{k=i+1}^n w_k$$

denoted \widehat{w}'_i is an admissible generalised reduced writing of w .

PROOF — Let us denote by w'_k for $k \in [1, N]$ the terms of the generalised writing. It is clear that it is reduced. Let us prove that it is admissible. Because the writing \widehat{w} is admissible, it is clear that the inclusion $P^{w'_{N-k}} \cap G_{w'_{N-k}} \subset P_{N-k}$ holds for $k \leq i + 2$. But for $k \geq i + 1$ then w'_{N-k} is a simple reflexion s_β and we have $G_{w'_{N-k}} = SL_2(\beta)$ and $P^{w'_{N-k}} \cap G_{w'_{N-k}}$ is contained in the Borel B so that the inclusion in P_{N-k+1} is trivial. \square

REMARK 5.8. — *Let us remark that the classical Bott-Samelson $\widetilde{X}(\widetilde{w})$ resolution is given by $\widehat{X}(\widehat{w}'_1)$.*

PROPOSITION 5.9. — *There is a morphism $\widehat{X}(\widehat{w}'_i) \rightarrow \widehat{X}(\widehat{w}'_{i+1})$ for all $i \in [1, n]$ (for $i = n$, let us fix $\widehat{X}(\widehat{w}'_{n+1}) = \widehat{X}(\widehat{w})$) and the morphism $\pi : \widetilde{X}(\widetilde{w}) \rightarrow X(w)$ from the Bott-Samelson resolution to the Schubert variety factors through these morphisms. In particular we will denote by $\widetilde{\pi}$ the morphism from $\widetilde{X}(\widetilde{w})$ to $\widehat{X}(\widehat{w})$.*

PROOF — The variety $\widehat{X}(\widehat{w}'_i)$ is the quotient of the product

$$\left(\prod_{k=1}^i \prod_{j=1}^{r_k} \overline{(P_{s_{j,k}} \cap G_{s_{j,k}}) s_{j,k} (P^{s_{j,k}} \cap G_{s_{j,k}})} \right) \times \prod_{k=i+1}^n \overline{(P_{w_k} \cap G_{w_k}) w_k (P^{w_k} \cap G_{w_k})}$$

by the product

$$\left(\prod_{k=1}^i \prod_{j=1}^{r_k} P^{s_{j,k}} \cap G_{s_{j,k}} \right) \times \prod_{k=i+1}^n P^{w_k} \cap G_{w_k}.$$

The action respects multiplication and in particular the multiplication map on the i^{th} factor

$$\prod_{j=1}^{r_i} \overline{(P_{s_{j,i}} \cap G_{s_{j,i}}) s_{j,i} (P^{s_{j,i}} \cap G_{s_{j,i}})} \rightarrow \overline{(P_{w_i} \cap G_{w_i}) w_i (P^{w_i} \cap G_{w_i})}$$

and the identity map on all the other factors is still defined modulo the action giving a map $\widehat{X}(\widehat{w}'_i) \rightarrow \widehat{X}(\widehat{w}'_{i+1})$. This map is simply the identity on all but one fibration (the one of fiber $X(w_i)$) and on this fibration is it given by the map $\widetilde{X}(\widetilde{w}_i) \rightarrow X(w_i)$ from the Bott-Samelson resolution to the Schubert variety.

The last affirmation is a simple consequence of the associativity of the product: the morphism from $\widetilde{X}(\widetilde{w}) = \widehat{X}(\widehat{w}'_1)$ is given by the product of all the terms and the factorisations are given by making the product in a certain order. \square

REMARK 5.10. — (i) The morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ is $P_{\widehat{X}(\widehat{w})}$ -equivariant and in particular if the generalised reduced decomposition \widehat{w} is good, it is P_w -equivariant.

(ii) Let us explain this construction with the quiver Q_w of w and the configuration variety. Denote by $m(Q_w)$ the set of maximal elements of Q_w for the partial order \preceq .

The classical morphism $\pi : \widetilde{X}(\widetilde{w}) \rightarrow X(w)$ is given by the projection from the configuration variety $\widetilde{X}(\widetilde{w}) \in \prod_{i \in Q_w} G/P_{\beta_i}$ on the product $\prod_{i \in m(Q_w)} G/P_{\beta_i}$.

To give a generalised decomposition \widetilde{w} of w is equivalent to give a partition of the vertices of the quiver by subquivers $(Q_{w_i})_{i \in [1, n]}$. Then the morphism from $\widetilde{X}(\widetilde{w})$ to $\widehat{X}(\widehat{w}'_i)$ is given by the projection on the product $\prod_{k=1}^i \prod_{j \in Q_{w_k}} G/P_{\beta_j} \times \prod_{k=i+1}^n \prod_{j \in m(Q_{w_k})} G/P_{\beta_j}$.

We can give a generalisation of fact 3.6 for the morphism $\widetilde{\pi}$. Recall that we have an admissible generalised reduced decomposition \widehat{w} given by $w = w_1 \cdots w_n$ and a reduced decomposition of each w_i giving a reduced writing \widetilde{w} of w given by $s_{\beta_1} \cdots s_{\beta_r}$. In terms of quiver, the quiver Q_w of the reduced writing \widetilde{w} has a partition by subquivers Q_{w_i} isomorphic to quivers of the elements w_i . We denote by $p_{\widehat{w}}(Q_w)$ the set of vertices of Q_w which are pics for the quiver Q_{w_i} to which they belong.

COROLLARY 5.11. — The variety Z_K is not contracted by $\widetilde{\pi} : \widetilde{X}(\widetilde{w}) \rightarrow \widehat{X}(\widehat{w})$ if and only if for each $j \in [1, n]$ the part of the subword $\prod_{i \in [1, r] \setminus K} s_i$ corresponding to a subword of w_j is reduced.

In particular, assume that all the w_i are minuscule elements, the group of divisors of $\widehat{X}(\widehat{w})$ has a basis given by $\widetilde{\pi}_*[Z_i]$ for $i \in p_{\widehat{w}}(Q_w)$ and the group of 1-cycles of $\widehat{X}(\widehat{w})$ has a basis indexed by $[1, n]$ given by $\widetilde{\pi}_*[C_i]$ for i the maximal vertex of the quiver Q_{w_i} for $i \in [1, n]$.

PROOF — This comes from the description fiberwise of the morphism from $\widetilde{X}(\widetilde{w})$ to $\widehat{X}(\widehat{w})$ and the lemma 3.6. To verify that the described elements form a basis, let us recall that $\widehat{X}(\widehat{w})$ is a tower of locally trivial fibration with fibers $X(w_i)$ for $i \in [1, n]$ so we know from the case of minuscule Schubert varieties that these elements form a basis. \square

5.4 Constructing generalised reduced decomposition

In this paragraph, we are going to give a way of constructing good generalised reduced decomposition of an element w in a product of minuscule elements $(w_i)_{i \in [1, n]}$.

DEFINITION 5.12. — *Let $A \subset p(Q_w)$ be a subset of the set of pic of Q_w , we denote by $Q(A_w)$ the full subquiver of Q_w containing the vertices i of Q_w such that $i \not\prec j$ for all $j \in p(Q_w) \setminus A$.*

It is different from Q_w as soon as A is different from $p(Q_w)$.

PROPOSITION 5.13. — *(i) Each connected component C of the quiver $Q_w(A)$ is isomorphic to the quiver of a minuscule Schubert variety and in particular has a unique maximal element $m(C)$ for the partial order \preccurlyeq .*

(ii) When A has a unique element then $Q_w(A)$ is connected.

(iii) The quiver $\widehat{Q}_w(A)$ obtained from Q_w by removing the vertices of $Q_w(A)$ is also the quiver of a minuscule Schubert variety.

(iv) The set $p(Q_w(A))$ is A and the set $p(\widehat{Q}_w(A))$ is $p(Q_w) \setminus A$.

PROOF — (i) (a) Let us prove that in any connected component C there is only one maximal element for the partial order \preccurlyeq . Let j_1 and j_2 be two such maximal elements. By connectivity, there exists a sequence of vertices $i_0 = j_1, i_1, \dots, i_n = j_2$ such that there is an arrow linking i_k and i_{k+1} (in one sense or another). Let us take a minimal such sequence (that is to say n is minimal) and let x be the smallest integer in $[0, n]$ such that $i_x \preccurlyeq i_{x-1}$ and $i_x \preccurlyeq i_{x+1}$. Such an element exists because j_1 and j_2 are maximal. By minimality of n we have $i_{x-1} \neq i_{x+1}$ are different and thanks to proposition 4.2 the vertex $s(i_x)$ exists. The arrows arriving to $s(i_x)$ come from $i_{x-1} \in C$, $i_{x+1} \in C$ and maybe from a third vertex $k \preccurlyeq i_x$ (and thus $k \in C$). The vertex $s(i_x)$ has to be in C . If we replace i_x by $s(i_x)$, we get a new sequence of length n but with i_{x-1} the first term such that $i_{x-1} \preccurlyeq i_{x-2}$ and $i_{x-1} \preccurlyeq i_x$. By induction we get a sequence of length n such that the smallest $x \in [0, n]$ with $i_x \preccurlyeq i_{x-1}$ and $i_x \preccurlyeq i_{x+1}$ is $x = 1$. This tells us that $s(i_1) \in C$ and $j_1 = i_0 \preccurlyeq s(i_1)$ in C which is a contradiction to the maximality of j_1 .

(b) Let us now prove that any connected component C of $Q(A)$ satisfies the conditions of 4.2. Let k be a vertex of C such that $s(k)$ does not exist or is not in C .

In the first case, this means that there is at most one arrow from k and denote by j the end vertex of this arrow. If j is not in C (or does not exist) then k is the maximal element of C . Otherwise j is in C and there is exactly one arrow from k in C .

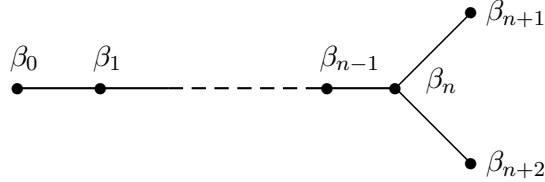
In the second case, we have two vertices k_1 and k_2 such that the arrows arriving to $s(k)$ come from k_1, k_2 and eventually a third one $k_3 \preccurlyeq k$ which has to be in C . As $s(k) \notin C$, at least one of the two vertices k_1 and k_2 has to be out of C . If both are out of C then k is the unique maximal element of C . Otherwise exactly one vertex from $\{k_1, k_2\}$ is in C .

If k is a vertex of C such that $s(k) \in C$, then we have two vertices k_1 and k_2 such that the arrows arriving to $s(k)$ come from k_1, k_2 and eventually a third one $k_3 \preccurlyeq k$ which has to be in C . These two elements have to be in C otherwise $s(k)$ would not be in C .

(c) We are left to prove that if $m(C)$ is the maximal element of C then $\beta(m(C))$ is a simple minuscule root for some group.

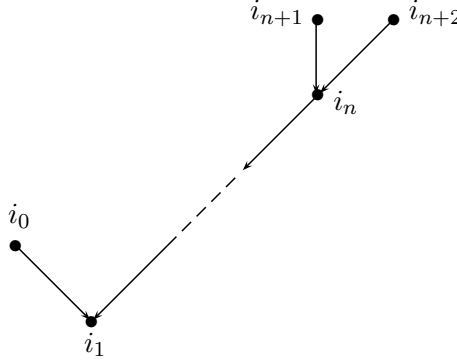
If the Dynkin diagram is of type A_n this is always true because any simple root is minuscule. Even if the set $\beta(C)$ of simple roots is of type A_n we are done. Let us assume that $\beta(C)$ contains a trivalent root γ and a root on each branch of the Dynkin diagram (remark that because C is connected, so is $\beta(C)$).

Denote by $m(C)$ the maximal element of C and by β the simple root $\beta(m(C))$. If this root was not a minuscule root of the Dynkin diagram $\beta(C)$ (a sub-Dynkin diagram of the one of G) then we would have the following situation:

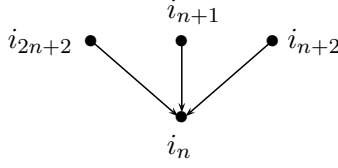


such that $\beta_1 = \beta$, $\beta_n = \gamma$ and all the simple roots β_i are in $\beta(C)$. There are two distinct simple roots β_{n+1} and β_{n+2} in $\beta(C)$ and not in $[\beta, \gamma]$ such that $\langle \gamma^\vee, \beta_{n+i} \rangle \neq 0$ for $i = 1, 2$. There is a simple root β_0 in $\beta(C)$ different from all the β_i for $i \in [1, n+2]$ such that $\langle \beta^\vee, \beta_0 \rangle \neq 0$.

Denote by i_k the biggest (for \preceq) vertex in C such that $\beta(i_k) = \beta_k$ for all $k \in [0, n+2]$. Then because C satisfies the properties of proposition 4.2, we see that in C , for all $k \in [2, n+1]$ there exists a unique arrow from i_k and it goes to i_{k-1} . In the same way there exists a unique arrow from i_{n+2} and it goes to i_n and from i_0 to $i_1 = m(C)$. This means that in C we have the following subquiver:



Now let us consider the subquiver Q' of Q corresponding to the vertices i such that $i \succ i_0$ or $i \succ i_0$ or $i \succ i_0$. This is a quiver corresponding to a minuscule Schubert variety. Each time there is a hole i in the quiver, we can add a new vertex j such that $\beta(j) = \beta(i)$ to obtain a quiver which still corresponds to a minuscule Schubert variety. We can thus add a vertex i_{n+3} with $\beta(i_{n+3}) = \beta$ and by induction vertices i_{n+2+k} with $\beta(i_{n+2+k}) = \beta(i_k)$ for all $k \in [1, n-1]$. In this new quiver we have the following subquiver:



But we also could have choosen $\beta(i_{2n+2}) = \beta(i_n)$ proving that in the quiver Q_ϖ of the minuscule homogeneous variety there exists a vertex j with $s(j) = i$. But then between j and i_n there would be three vertices (namely i_{n+1} , i_{n+2} and i_{2n+2}) having an arrow to i_n . This contradicts proposition 4.2 for Q_ϖ .

(ii) If A has a unique element then all vertices of $Q_w(A)$ are bigger than the unique element of A and $Q_w(A)$ is connected.

(iii) The quiver $\widehat{Q}_w(A)$ is obtained from Q_w by removing all the vertices smaller than $m(C)$ for any connected component C of $Q_w(A)$. It is thus (see proposition 4.5) the quiver of a minuscule Schubert variety.

(iv) Clear from the definitions. □

To construct a partition of the quiver Q_w of a minuscule element w into quivers $(Q_{w_i})_{i \in [1, n]}$ with w_i a minuscule elements, it suffices to give a partition $(A_i)_{i \in [1, n]}$ of the set $p(Q_w)$ of the pics of the quiver. Indeed, given such a partition $(A_i)_{i \in [1, n]}$, we define by induction a sequence $(Q_i)_{i \in [0, n]}$ of quivers with $Q_0 = Q$ and $Q_{i+1} = \widehat{Q}_i(A_{i+1})$ $Q_{w_1} = Q_w(A_1)$. We then denote by Q_{w_i} the quiver $Q_{i-1}(A_i)$. The quivers $(Q_{w_i})_{i \in [1, n]}$ form a partition of Q_w by quivers associated to minuscule elements w_i .

REMARK 5.14. — (i) For such partitions (giving a reduced generalised decomposition \widehat{w}), we have $p_{\widehat{w}}(Q_w) = p(Q_w)$.

(ii) For such constructions, the vertices of Q_{w_i} are the vertices x such that there exists a pic $p \in A_i$ with $p \preceq x$ and $p' \not\preceq x$ for any pic p' in A_j with $j > i$.

For such partitions $(Q_{w_i})_{i \in [1, n]}$ of the quiver Q_w coming from partitions $(A_i)_{i \in [1, n]}$ of $p(Q_w)$ we have a reduced generalised decomposition $w = w_1 \cdots w_n$ denoted \widehat{w} .

PROPOSITION 5.15. — *The reduced generalised writing \widehat{w} is good.*

PROOF — We have to prove that the inclusions $\Sigma(P_{w_{i+1} \cdots w_n}) \subset \Sigma(P^{w_i} \cap G_{w_i})$ and $\partial(G_{w_i}) \subset \Sigma(P_{w_i \cdots w_n})$. But the set $\Sigma(P_{w_i \cdots w_n})$ is the set $\beta(\{i \in Q_w / i \text{ is a hole of } Q_{w_i \cdots w_n}\})$ where $Q_{w_i \cdots w_n}$ is the subquiver of Q_w whose vertices are in $\cup_{k \geq i} Q_{w_k}$ (see proposition 4.11).

The set $\Sigma(P^{w_i} \cap G_{w_i})$ is the set $\beta(m_i) \cup \text{Supp}(w_i)^c$ where m_i is the maximal vertex of Q_{w_i} . So for the first inclusion we only have to prove that for any simple root $\beta \in \text{Supp}(w_i) \cap \Sigma(P_{w_{i+1} \cdots w_n})$ we have $\beta = \beta(m_i)$. But as $\beta \in \text{Supp}(w_i)$, there exists $j \in Q_{w_i}$ such that $\beta(j) = \beta$. Let j be the biggest such vertex. If j was not the biggest element m_i in Q_{w_i} then there would exist in Q_{w_i} an element k with an arrow from j to k . But then there are two cases, if $s(j)$ exists, it is a hole of $Q_{w_{i+1} \cdots w_n}$ and there are two vertices k_1 and k_2 having an arrow to $s(j)$ so between j

and $s(j)$ there are three vertices k , k_1 and k_2 and this is impossible thanks to proposition 4.2. If $s(j)$ does not exist, then j is a virtual hole of $Q_{w_{i+1}\dots w_n}$ and there is a vertex k' such that $\langle \beta(k'), \beta(j) \rangle \neq 0$ so that $s(j)$ does not exist but there are two vertices k and k' having an arrow coming from j and this is impossible thanks to proposition 4.2.

For the second inclusion, let β be a simple root in $\partial(G_{w_i})$, then there exists a vertex $j \in Q_{w_i}$ with $\langle \beta(j), \beta \rangle \neq 0$. If β is not in the support of $w_i \dots w_n$ then β is the simple root of a virtual hole and $\beta \in \Sigma(P_{w_i \dots w_n})$. If β is in this support then there exists a vertex k such that $\beta(k) = \beta$. Let k be the smallest such vertex, we have an arrow from j to k thus k is not a pic of $Q_{w_i \dots w_n}$ and thus not a pic of $Q_{w_{i+1} \dots w_n}$ (see the previous proposition). In particular there exists a vertex $x \in Q_{w_i \dots w_n}$ with an arrow from x to k . But then k is the smallest vertex with $\beta(k) = \beta$ in $Q_{w_i \dots w_n}$ and there are two arrows arriving to k thus it is a hole of $Q_{w_i \dots w_n}$ and we are done. \square

We now give here three types of partitions of Q_w constructed in this way.

CONSTRUCTION 1. — Choose any order $\{i_1, \dots, i_n\}$ on the set $p(Q_w)$ of the pics of Q_w and set $A_k = \{i_k\}$.

CONSTRUCTION 2. — Define a partition $(A_i)_{i \in [1, n]}$ by induction: A_1 is the set of pics with minimal height and A_{i+1} is the set of pics in $p(Q_w) \setminus \bigcup_{k=1}^i A_k$ with minimal height.

Before giving the last construction let us fix some notations and prove the following proposition. Recall that $p_{\widehat{w}}(Q_w) = p(Q_w)$ (with these constructions) is the set of all vertices j of Q_w such that there exists an integer $i \in [1, n]$ with $j \in p(Q_{w_i})$. Let us denote by $m_{\widehat{w}}(Q_w)$ the set of vertices j of Q_w such that j is a maximal element of Q_{w_i} for some $i \in [1, n]$.

The partial order \preceq induces a partial order on $m_{\widehat{w}}(Q_w)$. Let us finally prove the following:

PROPOSITION 5.16. — Let $i \in m_{\widehat{w}}(Q_w)$, there exists a unique minimal element $f(i)$ in $m_{\widehat{w}}(Q_w)$ for \preceq such that $i \preceq f(i)$.

PROOF — Let us prove the following

LEMMA 5.17. — Let j and k in $m_{\widehat{w}}(Q_w)$ such that there exists $x \in Q_w$ with $x \preceq j$ and $x \preceq k$. Then we have either $j \preceq k$ or $k \preceq j$.

PROOF — We proceed by induction on $a + b$ where a and b are the indexes in $[1, n]$ such that $j \in Q_{w_a}$ and $k \in Q_{w_b}$. Let x be a maximal element (for \preceq) such that $x \preceq j$ and $x \preceq k$ and suppose that x is different from j and k .

If there is only one arrow from x say going to a vertex y , then we must have $y \preceq j$ and $y \preceq k$ contradicting the maximality. Let y_1 and y_2 be the two arriving elements of the two arrows from x . If $\beta(y_1) = \beta(y_2)$ then $y_1 \preceq y_2$ (or the converse) and we have $y_1 \preceq j$ and $y_1 \preceq k$ contradicting the maximality. We thus have $\beta(y_1) \neq \beta(y_2)$ and $y_1 \preceq j$ but $y_1 \not\preceq k$ and $y_2 \not\preceq j$ but $y_2 \preceq k$. This also implies that $s(x)$ exists because y_1 and y_2 are connected to the biggest element r of the quiver and so the segments $[\beta(y_1), \beta(r)]$ and $[\beta(y_2), \beta(r)]$ are contained

in $\beta(\{z \in Q_w / z \succ x, z \neq x\})$ and thus $\beta(x)$ is in this set. So $s(x)$ exists and we have $s(x) \not\prec j$ and $s(x) \not\prec k$.

Now let c, d and e in $[1, n]$ such that $s(x) \in Q_{w_c}$, $j_1 \in Q_{w_d}$ and $j_2 \in Q_{w_e}$. We must have $c \geq d$ and $c \geq e$ because $s(x) \succ j_1, j_2$. We must also have $a \geq d$ and $b \geq e$. But if p is a pic in A_c such that $s(x) \succ p$, we must have $p \preccurlyeq j_1$ or $p \preccurlyeq j_2$ which implies (see remark 5.14) that $d \geq c$ or $e \geq c$. We thus have $c = d$ or $c = e$. Assume for example that $c = d$ and denote by m the maximal element of Q_{w_c} . If $c = d = a$ then $j_2 \preccurlyeq s(x) \preccurlyeq m = j$ and $j_2 \preccurlyeq k$ a contradiction to the maximality of x . So $c = d < a$, but we have $x \preccurlyeq m$ and $x \preccurlyeq j$ and by induction, we must have $m \preccurlyeq j$. Then we have $j_2 \preccurlyeq s(x) \preccurlyeq m \preccurlyeq j$ and $j_2 \preccurlyeq k$ one more time a contradiction to the maximality of x . \square

The preceding lemma proves that for $i \in m_{\widehat{w}}(Q_w)$ (and even for any $i \in Q_w$) the set $\{j \in m_{\widehat{w}}(Q_w) / j \succ i\}$ is totally ordered and thus there exists a minimal element $f(i)$. \square

We can now give the last construction which is a particular case of construction 1. Because in construction 1 there is a bijection between $p(Q_w)$ and $m_{\widehat{w}}(Q_w)$ we define thanks to the preceding proposition the function f on the set $p(Q_w)$ simply by the following: if $p \in p(Q_w)$ is in Q_{w_i} and if $m \in m_{\widehat{w}}(Q_w)$ is the maximal element of Q_{w_i} then $f(m)$ is the maximal element of some Q_{w_j} . There is only one pic q in Q_{w_j} and we define $f(p) = q$.

CONSTRUCTION 3. — *Choose an order $\{i_1, \dots, i_n\}$ on the set $p(Q_w)$ of the pics of Q_w such that $h(i_k) \leq h(f(i_k))$ for all $k \in [1, n-1]$ and set $A_k = \{i_k\}$.*

This choice on the order is equivalent to choose an order $\{i_1, \dots, i_n\}$ on the set $p(Q_w)$ of the pics of Q_w such that if i_k and i_{k+1} are adjacent in the quiver then $h(i_k) \leq h(i_{k+1})$.

These constructions may produce non connected subquivers Q_{w_i} but thanks to lemma 5.6 we may assume (replacing these quivers by their connected components) that all the quivers Q_{w_i} are quivers of minuscule Schubert varieties.

Construction 3 will give all relative minimal models of $X(w)$ and construction 2 will give the relative canonical model of $X(w)$.

6 Relative Mori theory of minuscule Schubert varieties

In this paragraph we prove the above assertion on the relative canonical and minimal models of a minuscule Schubert variety $X(w)$. We only consider generalised reduced writing \widehat{w} of w obtained thanks one of the three previous constructions.

6.1 Ample divisors and effective curves

Recall that we described a basis of divisors and 1-cycles on $\widehat{X}(\widehat{w})$ in the following way: the group of divisors of $\widehat{X}(\widehat{w})$ has a basis given by $D_i = \widetilde{\pi}_*[Z_i]$ for $i \in p_{\widehat{w}}(Q_w) = p(Q_w)$ and the group of 1-cycles of $\widehat{X}(\widehat{w})$ has a basis given by $\widetilde{\pi}_*[C_i]$ for $i \in m_{\widehat{w}}(Q_w)$. Recall also that (cf. [FMPSS])

the Chow groups of $\widehat{X}(\widehat{w})$ are generated by B -orbits, free over \mathbb{Z} and the Picard group is dual to the group of 1-cycles.

We have seen in remark 5.10 that the morphism $\tilde{\pi} : \tilde{X}(\tilde{w}) \rightarrow \widehat{X}(\widehat{w})$ is the projection from $\tilde{X}(\tilde{w})$ to the product $\prod_{i \in m_{\widehat{w}}(Q_w)} G/P_i$. We have on $\widehat{X}(\widehat{w})$ a projection $p_i : \widehat{X}(\widehat{w}) \rightarrow G/P_i$ for all $i \in m_{\widehat{w}}(Q_w)$. Let us define the Cartier divisor $\mathcal{M}_i = p_i^*(\mathcal{O}_{G/P_i}(1))$ for all $i \in m_{\widehat{w}}(Q_w)$. We have $\mathbf{L}_i = \tilde{\pi}^* \mathcal{M}_i$ for all $i \in m_{\widehat{w}}(Q_w)$. Because of the description of $\widehat{X}(\widehat{w})$ as a tower of locally trivial fibration with fibers isomorphic to minuscule Schubert varieties $X(w_i)$, we have the following:

FACT 6.1. — *The family $(\mathcal{M}_i)_{i \in m_{\widehat{w}}(Q_w)}$ is a basis of $\text{Pic}(\widehat{X}(\widehat{w}))$.*

Recall that we gave a basis $(Y_i)_{i \in [1, r]}$ of the cone of effective 1-cycles $\tilde{X}(w)$ in paragraph 3.3.2. Because $[Y_i] = [C_i] - [C_{s(i)}]$ we see that we have the following:

FACT 6.2. — *The family $(\tilde{\pi}_*[Y_i])_{i \in m_{\widehat{w}}(Q_w)}$ is a basis of the group of 1-cycles on $\widehat{X}(\widehat{w})$ and is dual to the basis $(\mathcal{M}_i)_{i \in m_{\widehat{w}}(Q_w)}$.*

PROOF — The pull-back of \mathcal{M}_i by $\tilde{\pi}$ is \mathbf{L}_i and we have $\mathbf{L}_i \cdot [Y_j] = \delta_{i,j}$ so by projection formula we get the result. \square

PROPOSITION 6.3. — *The family $([\tilde{\pi}_* Y_i])_{i \in m_{\widehat{w}}(Q_w)}$ is a basis of the cone of effective classes and the family $(\mathcal{M}_i)_{i \in m_{\widehat{w}}(Q_w)}$ is a basis of the closure of the ample cone.*

PROOF — The embedding of $\widehat{X}(\widehat{w})$ in $\prod_{i \in m_{\widehat{w}}(Q_w)} G/P_i$ is given by $\bigotimes_{i \in m_{\widehat{w}}(Q_w)} \mathcal{M}_i$. The cone generated by the \mathcal{M}_i is thus contained in the ample cone.

Conversely, let A be an ample sheaf and let $a_i = A \cdot [\tilde{\pi}_* Y_i]$ for $i \in m_{\widehat{w}}(Q_w)$. We must have $a_i > 0$ and the divisor $A - \sum_i a_i \mathcal{M}_i$ is numerically trivial and we get the result.

By duality we have the result on curves. \square

Proposition 4.13 has the following relative version.

PROPOSITION 6.4. — *We have the formula*

$$\mathcal{M}_i = \sum_{k \in p(Q_w), k \preccurlyeq i} D_k.$$

PROOF — Because the variety $\widehat{X}(\widehat{w})$ is normal with rational singularities, we have $\mathcal{M}_i = \tilde{\pi}_* \mathbf{L}_i$ and we obtain the formula in the same way as in proposition 4.13 thanks to the fact that all quivers Q_{w_i} are associated to minuscule Schubert varieties. \square

Let us generalise corollary 4.14 and give a criterion for a variety $\widehat{X}(\widehat{w})$ to be locally factorial.

COROLLARY 6.5. — *The variety $\widehat{X}(\widehat{w})$ is locally factorial if and only if for all $i \in [1, n]$ the quiver Q_{w_i} has a unique pic.*

6.2 Canonical divisor of $\widehat{X}(\widehat{w})$

As in proposition 4.15, we have $K_{\widehat{X}(\widehat{w})} = \widetilde{\pi}_* K_{\widetilde{X}(\widetilde{w})}$. The same calculus gives the:

FACT 6.6. — *We have*

$$-K_{\widehat{X}(\widehat{w})} = \sum_{k \in p(Q_w)} (h(k) + 1) D_k.$$

For the three constructions, the pics of a fixed quiver Q_{w_i} have all the same height so we can define $h(w_i)$ to be the height of any pic of Q_{w_i} . Set $h(w_{n+1}) = -1$. Proposition 6.4 gives us the following (by induction on the order of $m_{\widehat{w}}(Q_w)$):

COROLLARY 6.7. — *We have the formula*

$$-K_{\widehat{X}(\widehat{w})} = \sum_{i \in m_{\widehat{w}}(Q_w)} (h(w_i) - h(w_{f(i)})) \mathcal{M}_i$$

and in particular $\widehat{X}(\widehat{w})$ is Gorenstein.

6.3 Types of singularities

In this paragraph we are going to prove that the variety $\widehat{X}(\widehat{w})$ has terminal singularities in case of constructions 1, 2 and 3.

For this we use the resolution $\widetilde{X}(\widetilde{w})$ of $\widehat{X}(\widehat{w})$ and compare the canonical divisor $K_{\widetilde{X}(\widetilde{w})}$ to the pull-back of the canonical divisor $K_{\widehat{X}(\widehat{w})}$ by $\widetilde{\pi}$. We need the following fact coming directly from the formula of paragraph 3.3.1 and lemma 4.16:

FACT 6.8. — *We have the formula*

$$-K_{\widetilde{X}(\widetilde{w})} = \sum_{i=1}^r (h(i) + 1) \xi_i.$$

We can now prove the following:

PROPOSITION 6.9. — *The variety $\widehat{X}(\widehat{w})$ has terminal (and hence canonical) singularities.*

PROOF — Let us calculate the pull-back of $K_{\widehat{X}(\widehat{w})}$ by $\widetilde{\pi}$:

$$-\widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} = \sum_{i \in m_{\widehat{w}}(Q_w)} (h(w_i) - h(w_{f(i)})) \mathbf{L}_i$$

but thanks to proposition 3.14 and the fact that the quivers Q_{w_i} are the quivers of minuscule Schubert varieties we have

$$\mathbf{L}_i = \sum_{k \prec i} \xi_k$$

giving

$$-\widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} = \sum_{i=1}^n \left((h(w_i) + 1) \sum_{k \in Q_{w_i}} \xi_k \right)$$

We get for the difference:

$$K_{\widehat{X}(\widehat{w})} - \widetilde{\pi}^* K_{\widehat{X}(\widehat{w})} = \sum_{i=1}^n \sum_{k \in Q_{w_i}} (h(w_i) - h(k)) \xi_k.$$

But $h(w_i)$ is the highest height of an element in Q_{w_i} so $h(w_i) - h(k) \geq 0$ with equality for k a pic of the quiver that is to say if and only if ξ_i is not contracted by $\widetilde{\pi}$. \square

6.4 Description of the relative minimal and canonical models

We can now prove our results on relative minimal and canonical models of minuscule Schubert varieties.

THEOREM 6.10. — (i) *The varieties $\widehat{X}(\widehat{w})$ obtained from construction 3 are relative minimal models of $X(w)$.*

(ii) *The variety $\widehat{X}(\widehat{w})$ obtained from construction 2 is the relative canonical model of $X(w)$.*

PROOF — (i) We have to prove that any curve C contracted by $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ satisfies $[C] \cdot K_{\widehat{X}(\widehat{w})} \geq 0$. The class $[C]$ can be written as $[C] = \sum_i a_i [\widetilde{\pi}_* Y_i]$ with $a_i \geq 0$ and $a_n = 0$ (because the curve is contracted). We just have to prove the non negativity of the intersections $K_{\widehat{X}(\widehat{w})} \cdot [\widetilde{\pi}_* Y_j]$ for $j \in [1, n-1]$. We have $K_{\widehat{X}(\widehat{w})} \cdot [\widetilde{\pi}_* Y_j] = h(w_{f(j)}) - h(w_j)$ and by construction 3 this intersection is non negative.

(ii) It suffices to prove that the contracted curves have a positive intersection with the canonical divisor and this comes from the previous calculus and construction 2. \square

Let $\widehat{X}(\widehat{w})$ a variety obtained from one of the construction 1, 2 or 3. We can completely describe the extremal rays of the relative cone of effective 1-cycles on $\widehat{X}(\widehat{w})$ (i.e. the effective 1-cycles contracted by $\widehat{\pi}$).

FACT 6.11. — (i) *The extremal rays of $\widehat{X}(\widehat{w})$ are given by the classes $[\widetilde{\pi}_* Y_j]$ such that $h(w_{f(j)}) < h(w_j)$.*

(ii) *If $\widehat{X}(\widehat{w})$ is obtained from construction 3, then there is no extremal ray. However, if D is any effective divisor, then the $(K_{\widehat{X}(\widehat{w})} + D)$ -extremal rays are given by the classes $[\widetilde{\pi}_* Y_j]$ such that $(K_{\widehat{X}(\widehat{w})} + D) \cdot [\widetilde{\pi}_* Y_j] < 0$.*

PROOF — (i) Let $[C]$ the class of an effective curve. Then there exists non negative integers a_i such that $[C] = \sum_i a_i [\widetilde{\pi}_* Y_i]$. Denote by μ_j (resp. ν_k and ω_l) the classes $[\widetilde{\pi}_* Y_i]$ such that $K_{\widehat{X}(\widehat{w})} \cdot [\widetilde{\pi}_* Y_i] < 0$ (resp. > 0 and $= 0$). For each j and k there is a linear combination with positive coefficient $x_{j,k} \mu_j + y_{j,k} \nu_k$ such that $K_{\widehat{X}(\widehat{w})} \cdot (x_{j,k} \mu_j + y_{j,k} \nu_k) = 0$. It is easy to check that if $K_{\widehat{X}(\widehat{w})} \cdot [C] < 0$ then $[C]$ has to be a linear combination with non negative coefficient of classes (μ_j) , $(x_{j,k} \mu_j + y_{j,k} \nu_k)$ and (ω_l) proving the result.

(ii) The same proof with $K_{\widehat{X}(\widehat{w})} + D$ instead of $K_{\widehat{X}(\widehat{w})}$ works. It works for all varieties $\widehat{X}(\widehat{w})$ obtained from construction 1, 2 or 3. \square

Let us consider a fixed order (p_1, \dots, p_n) on the pics of the quiver Q_w and \widehat{w} the good reduced generalised writing it induces and let us denote by \widehat{w}' the good reduced generalised writing induced by the order (q_1, \dots, q_n) on the pics where $q_k = p_k$ for $k \notin \{i, i+1\}$, $q_i = p_{i+1}$ and $q_{i+1} = q_i$. Denote by \widehat{w}'' the good reduced generalised writing given by $w_k'' = w_k$ for $k < i$, $w_i'' = w_i w_{i+1}$ and $w_k = w_{k+1}$ for $k > i$ that is to say obtained by the partition $(A_k)_{k \in [1, n-1]}$ of $p(Q_w)$ given by $A_k = \{p_k\}$ for $k < i$, $A_i = \{p_i, p_{i+1}\}$ and $A_k = \{p_{k+1}\}$ for $k > i$. Denote by k_i (resp. k_{i+1}) the maximal vertex of Q_{w_i} (resp. $Q_{w_{i+1}}$).

We have two morphisms (for example because of remark 5.10)

$$f : \widehat{X}(\widehat{w}) \rightarrow \widehat{X}(\widehat{w}'') \quad \text{and} \quad f' : \widehat{X}(\widehat{w}') \rightarrow \widehat{X}(\widehat{w}'').$$

PROPOSITION 6.12. — *Let $\widehat{X}(\widehat{w})$ obtained from construction 1, 2 or 3.*

(i) *If $k_i \not\preceq k_{i+1}$ (i.e. if $i+1 \neq f(i)$) then the morphisms f and f' are isomorphisms.*

(ii) *If $f(i) = i+1$ and $\tilde{\pi}_*[Y_{k_i}] \cdot K_X > 0$, then f (resp. f') is the small contraction corresponding to the extremal ray $\mathbb{R}\tilde{\pi}_*[Y_{k_i}]$ (resp. $\mathbb{R}\tilde{\pi}_*[Y_{k'_i}]$) and f is the flip of f' .*

(iii) *If $f(i) = i+1$ and $\tilde{\pi}_*[Y_{k_i}] \cdot K_X = 0$, denote $D = D_{i+1}$ (resp. $D' = D'_i$) then for $\varepsilon > 0$, the class $\tilde{\pi}_*[Y_{k_i}]$ (resp. $\tilde{\pi}_*[Y_{k'_i}]$) is extremal for $K_{\widehat{X}(\widehat{w})} + \varepsilon D$ (resp. $K_{\widehat{X}(\widehat{w}')} + \varepsilon D'$) and f (resp. f') is the small contraction corresponding to the extremal ray $\mathbb{R}\tilde{\pi}_*[Y_{k_i}]$ (resp. $\mathbb{R}\tilde{\pi}_*[Y_{k'_i}]$). The morphism (f', D') is the flop of (f, D) .*

PROOF — (i) Because k_i and k_{i+1} are non comparable for \preceq , we have thanks to lemma 5.17 that w_i and w_{i+1} satisfy the hypothesis of lemma 5.6 and we have the result.

In the cases where $f(i) = i+1$, we already know that $\widehat{X}(\widehat{w})$ and $\widehat{X}(\widehat{w}')$ are locally factorial with terminal singularities. We also know that $\widehat{X}(\widehat{w}'')$ is normal and the morphisms are birational and mori-small (the group of Weil divisors has a basis given by the pics). Because of our description of Picard groups we also have $\rho(X/X'') = \rho(X'/X'') = 1$.

(ii) We are left to study the divisors $K_{\widehat{X}(\widehat{w})}$ and $K_{\widehat{X}(\widehat{w}')}$ on the fibers of f and f' . We will not describe these fibers in details in this proposition but more details will be given in the next paragraph. Because all the sheaves \mathcal{M}_{k_j} for $j \neq i$ are already defined on $\widehat{X}(\widehat{w}'')$ they are trivial on the fibers of f and the sheaf \mathcal{M}_{k_i} on $\widehat{X}(\widehat{w})$ is relatively ample with respect to f . The restriction of $K_{\widehat{X}(\widehat{w})}$ to the fibers of f is given by $(K_{\widehat{X}(\widehat{w})} \cdot \tilde{\pi}_*[Y_{k_i}])\mathcal{M}_{k_i}$ so that $K_{\widehat{X}(\widehat{w})}$ is ample, anti-ample or trivial according to the positivity of the intersection $K_{\widehat{X}(\widehat{w})} \cdot \tilde{\pi}_*[Y_{k_i}]$ and thus according to the height of the pics. This proves in case (ii) that $K_{\widehat{X}(\widehat{w})}$ is f -ample.

Furthermore, the fiber of f is contained in $G/P_{\beta(k_i)}$ thus the classes of contracted curves are proportional to $\tilde{\pi}_*[Y_{k_i}]$.

In the same way we get that $-K_{\widehat{X}(\widehat{w}')}$ is f' -ample and the result.

(iii) Proposition 6.4 tel us that $D = D_{i+1}$ satisfies $\tilde{\pi}_*[Y_{k_i}] \cdot D < 0$ so the class $\tilde{\pi}_*[Y_{k_i}]$ is extremal for $K_{\widehat{X}(\widehat{w})} + \varepsilon D$.

The Bott-Samelson variety $\widetilde{X}(\widetilde{w})$ is a resolution of the birational morphism between $\widehat{X}(\widehat{w})$ and $\widehat{X}(\widehat{w}')$ and D is the image of $\xi_{p_{i+1}}$ whose image in $\widehat{X}(\widehat{w}')$ is D' so that D' is the strict transform of D .

We are left to study the divisors $K_{\widehat{X}(\widehat{w})}$, $K_{\widehat{X}(\widehat{w}')}$, D and D' on the fibers of the morphisms f and f' . The calculus in (1) proves the triviality of $K_{\widehat{X}(\widehat{w})}$ and $K_{\widehat{X}(\widehat{w}')}$ on the fibers. For D and D' , the same argument as for the canonical sheaves proves that their restriction to the fibers of f (resp. f') is a positive multiple of $-\mathcal{M}_{k_i}$ (resp. $\mathcal{M}_{k'_i}$) concluding the proof. \square

REMARK 6.13. — *If $f(i) = i + 1$ and $[Y_{k_i}] \cdot K_{\widehat{X}(\widehat{w})} < 0$ then by symmetry f' is the flip of f .*

COROLLARY 6.14. — *(i) The varieties $\widehat{X}(\widehat{w})$ obtained from construction 1 are linked by flips and flops and any variety obtained from $\widehat{X}(\widehat{w})$ by flips and flops comes from this construction.*

(ii) The varieties $\widehat{X}(\widehat{w})$ obtained from construction 3 are linked by flops and any variety obtained from $\widehat{X}(\widehat{w})$ by flops comes from this construction.

PROOF — (i) We know that the extremal rays (or more generally the $(K_{\widehat{X}(\widehat{w})} + D)$ -extremal rays) of $\widehat{X}(\widehat{w})$ are generated by the classes $\widetilde{\pi}_*[Y_{k_i}]$ with $K_{\widehat{X}(\widehat{w})} \cdot \widetilde{\pi}_*[Y_{k_i}] < 0$ (resp. $(K_{\widehat{X}(\widehat{w})} + D) \cdot \widetilde{\pi}_*[Y_{k_i}] < 0$). But the associated flip or flop gives a variety obtained by construction 1.

(ii) The same argument works in this case because $h(w_{f(i)}) = h(w_i)$ and we stay in the class of varieties obtained from construction 3. \square

REMARK 6.15. — *For any variety obtained from construction 1 we proved existence and termination of flips and flops (in the sens of K. Matsuki [Mat]).*

COROLLARY 6.16. — *The relative minimal models of $X(w)$ are exactly the varieties obtained from construction 3.*

PROOF — We use theorem 12-1-8 of [Mat], the fact that varieties obtained from construction 3 are relative minimal models and existence and termination of flops for these varieties. \square

7 Small IH -resolutions of minuscule Schubert varieties

In this section we prove that the morphism $\widehat{X}(\widehat{w}) \rightarrow X(w)$ obtained from construction 3 is IH -small. We then discuss the smoothness of $\widehat{X}(\widehat{w})$ and describe all IH -small resolutions of minuscule Schubert varieties. Let us first recall the definition of an IH -small morphism:

DEFINITION 7.1. — *A morphism $\pi : Y \rightarrow X$ is said to be IH -small if for all $k > 0$, we have*

$$\text{codim}_X \{x \in X \mid \dim(\pi^{-1}(x)) = k\} > 2k.$$

A small morphism $\pi : Y \rightarrow X$ is a small resolution of X if Y is smooth.

In this section we will use a case by case analysis instead of a global proof for all minuscule Schubert varieties in the same time. A combinatorial direct proof on the quiver is possible but it would lead to a too complicated combinatorial discussion and we prefer avoiding it.

7.1 Necessary condition

Let us first prove the following proposition showing that among the morphisms $\pi : \widehat{X}(\widehat{w}) \rightarrow X(w)$ obtained from construction 1 only the one coming from construction 3 can be small:

PROPOSITION 7.2. — *Let $\widehat{X}(\widehat{w})$ obtained from construction 1 but not from construction 3. Then the morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ is not IH -small.*

PROOF — If $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ was small then, because $\widehat{X}(\widehat{w})$ has terminal singularities, it would be a relative minimal model (this is a consequence of the proof by B. Totaro [To] of theorem 7.9). This is not the case by corollary 6.16.

One can give an explicit subvariety in $X(w)$ not satisfying the IH -small condition for $\widehat{\pi}$. Let i be a pic of Q_w such that $h(f(i)) < h(i)$ (such a pic exists because the resolution is not obtained from construction 3). Let us consider the smallest (for \preceq) vertex $j \in Q_w$ such that $j \succ i$ and $j \succ f(i)$. Then one can prove that the image $\widetilde{\pi}(Z_k)$ of the divisor $Z_j \subset \widetilde{X}(\widetilde{w})$ in $\widehat{X}(\widehat{w})$ is of codimension $h(f(i)) - h(j) + 1$ and that its image $\pi(Z_j)$ in $X(w)$ is of codimension $h(i) - h(j) + h(f(i)) - h(j) + 1$. The fiber above $\pi(Z_j)$ contains $\widetilde{\pi}(Z_j)$ and is of dimension at least $h(i) - h(j)$. But we have

$$\text{Codim}_{X(w)}(\pi(Z_j)) = h(i) - h(j) + h(f(i)) - h(j) + 1 \leq 2(h(i) - h(j)).$$

□

On the contrary when we choose a good order on the pics then we obtain the following

THEOREM 7.3. — *The morphisms $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ obtained from construction 3 are IH -small.*

In subsections 7.2, 7.3 and 7.4 we are going to give a proof of this theorem. Let us first give an easy corollary:

COROLLARY 7.4. — *The morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ obtained from construction 2 is IH -small.*

PROOF — Indeed, any morphism $\widehat{\pi}'$ obtained from construction 3 factors through the morphism $\widehat{\pi}$ and it is easy to verify that this implies that, as $\widehat{\pi}'$ is IH -small, $\widehat{\pi}$ is IH -small. □

7.2 Fibers

We will adapt the technics of [SV1] in our setting. The idea that choosing a good order in the pics will produce IH -small morphisms comes from A. Zelevinsky's paper [Ze].

Let us recall that the varieties $\widehat{X}(\widehat{w})$ were constructed by induction. The first step being given by the morphism $p : \overline{PuQ} \times^Q X(v) \rightarrow X(w)$ where P is the stabiliser of $X(w)$, v is obtained from w by removing the first pic and u corresponds to the removed vertices. The group Q is the intersection of P with the stabiliser of $X(v)$. By induction there exists a resolution

$\pi' : \widehat{X}(\widehat{v}) \rightarrow X(v)$ equivariant under the stabiliser of $X(v)$. The resolution is given by the fiber product $\widehat{\pi} : \widehat{X}(\widehat{w}) = \overline{PuQ} \times^Q \widehat{X}(\widehat{v}) \rightarrow X(w)$.

Let us prove a lemma giving a description of the fibers of π and a formula on their dimension. This lemma is directly inspired by lemma 2.1 of [SV1]. If $w' \leq w$ in the Bruhat order, let us denote by $U(w')$ the P -orbit (P is the stabiliser of $X(w)$) of $e_{w'}$ (the fixed point of the torus corresponding to the Schubert cell of w') in $X(w)$. Because π is P -equivariant, all the fibers of points in $U(e_{w'})$ are isomorphic and to calculate $f_{\pi, w'}$ the dimension of the fiber $\pi^{-1}(e_{w'})$, it is enough to calculate $\dim(\pi^{-1}(U(e_{w'}))) - \dim(U(e_{w'}))$.

LEMMA 7.5. — *Define the set*

$$S(w', w) = \left\{ (u', v') \in W \left/ \begin{array}{l} u' \leq u \text{ and } v' \leq v, \text{ in the Bruhat order and} \\ PX(u'v') = X(w'), PX(u') = X(u') \text{ and } QX(v') = X(v') \end{array} \right. \right\}.$$

$$(i) \text{ We have } p^{-1}(U(w')) = \bigcup_{(u', v') \in S(w', w)} Pu'Q \times^Q Qe_{v'}.$$

$$(ii) \text{ We have } \pi^{-1}(U(w')) = \bigcup_{(u', v') \in S(w', w)} Pu'Q \times^Q \pi'^{-1}(Qe_{v'}).$$

(iii) This gives the formula

$$f_{\pi, w'} = \text{Card}(Q_{u'}) + f_{\pi', v'} + \text{Card}(Q_{v'}) - \text{Card}(Q_{w'}) = \text{Card}(Q_{u'}) + f_{\pi', v'} - \text{Codim}_{X(w)}(X(v'))$$

for some $(u', v') \in S(w', w)$.

PROOF — (i) Let $(u', v') \in S(w', w)$, we have the inclusions:

$$p(Pu'Q \times^Q Qe_{v'}) \subset Pe_{u'}Qe_{v'} \subset Pe_{u'}X(v') \subset PX(u'v') = X(w').$$

Furthermore $p(Pu'Q \times^Q Qe_{v'})$ is a P -orbit and contains $Pe_{u'}e_{v'} = Pe_{w'}$ so $Pu'Q \times^Q Qe_{v'}$ is contained in $p^{-1}(U(w'))$.

Conversely, if $(x, y) \in \overline{PuQ} \times^Q X(v)$ is such that $p(x, y) = xy \in U(w')$, then there are elements u' and v' in the Weyl group such that we have $PX(u') = X(u')$, $QX(v') = X(v')$ and $\overline{PxQ} \times^Q \overline{QyP} / P_{\varpi} = \overline{Pu'Q} \times^Q X(v')$. But then there exists $(p, q) \in P \times Q$ such that $pe_{u'} = x$ and $qe_{v'} = y$ so that we have $(x, y) \in Pu'Q \times^Q Qe_{v'}$. Furthermore, there exists $p' \in P$ such that $p'xy = e_{w'}$ thus $p'pe_{u'}qe_{v'} = e_{w'}$. This implies, because $QX(v') = X(v')$, that $PX(u'v') = X(w')$.

(ii) Comes directly from (i).

(iii) Because $\dim U(w') = \text{Card}(Q_{w'})$, we only need to prove that

$$\dim(\pi^{-1}(U(w'))) = \text{Card}(Q_{u'}) + f_{\pi', v'} + \text{Card}(Q_{v'})$$

for some $(u', v') \in S(w', w)$, it is true thanks to (ii). \square

REMARK 7.6. — (i) In the case where \overline{PuQ}/Q is an homogeneous variety, we recover lemma 2.1 of [SV1]. In this case we must have $u' = u$ because $X(u)$ is the only P -stable Schubert subvariety of $X(u)$. We then have $\text{Card}(Q_{u'}) = \text{Codim}_{X(w)}(X(v)) = \text{Card}(Q_w) - \text{Card}(Q_v)$.

(u) More generally, the Schubert variety $X(u')$ is a Schubert subvariety of $X(u)$ with the same stabiliser so that if i is a hole of its quiver then $\beta(i) \in \beta(t(Q_u))$. The Schubert variety $X(v')$ is a Schubert subvariety of $X(v)$ with stabiliser $\text{Stab}(w) \cap \text{Stab}(v)$. If i is a hole of $Q_{v'}$ then $\beta(i)$ has to be in the union $\beta(t(Q_v)) \cup \beta(t(Q_w))$.

Let us now describe the condition $PX(u'v') = X(w')$. The quiver of the Schubert variety $X(u'v')$ is obtained by gluing the quiver of $X(u')$ above the quiver of $X(v')$. Furthermore, the quiver of the Schubert variety $PX(a)$ if $X(a)$ is a Schubert subvariety of $X(w)$, is the smallest subquiver Q of Q_w containing the quiver Q_a and such that $\beta(t(Q)) \subset \beta(t(Q_w))$. In particular, if we denote by A the set of vertices $i \in Q_a$ such that i is not the successor of an element of Q_a and $\beta(i) \in \beta(t(Q_w))$ then the set non virtual holes of $PX(a)$ is A and the virtual holes are associated to simple roots in $\beta(t(Q_w)) \setminus \beta(A)$.

If i is a hole of $Q_{w'}$ such that $\beta(i)$ is not in the support of u . Let j be the smallest vertex in $Q_{w'}$ such that $\beta(j) = \beta(i)$. Then j will be a vertex of $Q_{v'u'}$ with no predecessor and has to be a hole of $PX(u'v') = X(w')$. We must thus have $j = i$. In particular all the holes of $Q_{w'}$ associated to simple roots not in the support of u have to be holes of v' .

The proof of the theorem 7.3 will go as follows. Because the morphism π is P -equivariant where P is the stabiliser of $X(w)$, we need to prove that for any $w' \in W$ such that $X(w')$ is stable under P in $X(w)$, we have $\text{Codim}_{X(w)}(X(w')) > 2f_{\pi, w'}$.

For the classical cases (A_n and D_n) we will introduce two functions Γ and q such that

$$\text{Codim}_{X(w)}(X(w')) = \Gamma(w', w) + q(w', w)$$

The function q will take only non negative values and will be positive if $w' \neq w$.

We then proceed by induction on the number of pics of w (or on the number of fibrations in $\hat{X}(\hat{w})$) and prove the more stronger result:

$$\Gamma(w', w) \geq 2f_{\pi, w'}.$$

Because of the previous lemma, it is enough to prove that for all $(u', v') \in S(w', w)$ we have

$$\Gamma(w', w) \geq 2(\text{Card}(Q_{u'}) + f_{\pi', v'} - \text{Codim}_{X(w')}(X(v'))).$$

Let $\theta \in W$ such that $X(\theta)$ is the closure of the orbit of $X(v')$ in $X(v)$ under $\text{Stab}(X(v))$. We have $f_{\pi', v'} = f_{\pi', \theta}$ and by induction hypothesis we have $2f_{\pi', \theta} \leq \Gamma(\theta, v)$. We are thus reduced to prove:

$$2(\text{Card}(Q_{u'}) - \text{Codim}_{X(w')}(X(v'))) \leq \Gamma(w', w) - \Gamma(\theta, v).$$

We prove this formula in the following paragraph in the A_n and D_n case.

7.3 The case of A_n and D_n

To prove the result on smallness, we will need to describe the elements of $S(w', w)$ and calculated the dimension in the formula of lemma 7.5. The only two difficult cases of minuscule

Schubert varieties will be the cases of grassmannians (the varieties constructed are the varieties of Zelevinsky [Ze]) and of maximal isotropic subspaces in an even dimensional vector space endowed with a non degenerate quadratic form (some of these cases have been treated in [SV1] and we complete their study). Indeed the other minuscule Schubert varieties are those of E_6 and E_7 for which we will make a case by case analysis (which is tedious by not hard with lemma 7.5) and the case of Schubert varieties in a quadric which are very simple.

7.3.1 The A_n case

Let us consider a minuscule quiver Q_w of a Schubert variety in the grassmannian $\mathbb{G}(p, q)$ of p -dimensional subvector spaces of a q -dimensional vector space. We may assume that all the simple roots are in the support of w (otherwise we simply restrict the group) so that there will be no virtual hole in Q_w . The set of simple roots $\beta(t(Q_w))$ can be written as $\{\alpha_{k_1}, \dots, \alpha_{k_s}\}$ in the notation of [Bou]. Let us denote by t_1, \dots, t_s the holes such that $\beta(t_i) = \alpha_{k_i}$. Because of proposition 4.2, for all $i \in [2, s]$ there exists exactly one pic p_i between the holes t_{i-1} and t_i . Furthermore there must be a pic p_1 (resp. p_{s+1}) with $\beta(p_1) = \alpha_k$ (resp $\beta(p_{s+1}) = \alpha_k$) with $k < k_1$ (resp. $k > k_s$). In particular we see that the number n of pics equals $s + 1$.

Let us now define the following sequences $(a_i(w))_{i \in [1, s+1]}$ and $(b_i(w))_{i \in [0, s]}$ of integers (we will sometimes simply denote them by a_i and b_i omiting w):

$$\begin{cases} a_i(w) = h(p_i) - h(t_i) \text{ and } b_i(w) = h(p_{i+1}) - h(t_i) \text{ for } i \in [1, s], \\ a_{s+1}(w) = p - \sum_{i=1}^s a_i(w), \\ b_0(w) = q - p - \sum_{i=1}^s b_i(w). \end{cases}$$

It is an easy game on the quiver and the description of configuration varieties to verify that if the sequences of integers associated to the quiver Q_w are $(a_i(w))_{i \in [1, s+1]}$ and $(b_i(w))_{i \in [0, s]}$ then we have:

$$X(w) = \{V \in \mathbb{G}(p, q) / \dim(V \cap \mathbb{K}^{n_i}) \geq m_i \text{ for all } i \in [1, s]\}$$

where $n_i = \sum_{k=1}^i (a_k + b_{k-1})$ and $m_i = \sum_{k=1}^i a_k$.

Let $X(w')$ be a Schubert subvariety of $X(w)$ with the same stabiliser. Then we must have $\beta(t(Q_{w'})) \subset \beta(t(Q_w))$. For any hole t_i of Q_w let us define the depth of w' in t_i to be the integer

$$c_i = \text{Card}\{j \in Q_w \setminus Q_{w'} / \beta(j) = \beta(t_i)\}.$$

The same game on the quiver and the description of configuration varieties shows that the associated sequences are given by

$$\begin{cases} a_i(w') = a_i(w) + c_i - c_{i-1} \text{ for all } i \in [1, s+1] \\ b_i(w') = b_i(w) + c_i - c_{i+1} \text{ for all } i \in [0, s] \end{cases}$$

with $c_0 = c_{s+1} = 0$. We have

$$X(w) = \{V \in \mathbb{G}(p, q) / \dim(V \cap \mathbb{K}^{n_i}) \geq m_i + l_i \text{ for all } i \in [1, s]\}$$

with $l_i = \sum_{k=1}^i c_k$. These description enables us to give the following fact to calculate the codimension of $X(w')$ in $X(w)$ (we set $c_0 = c_{s+1} = 0$):

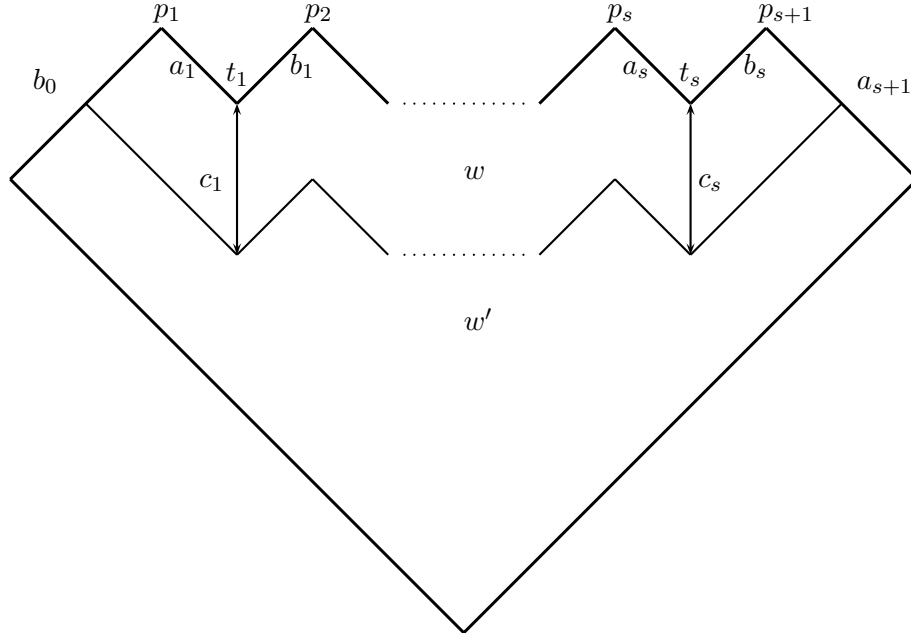
FACT 7.7. — *We have the formula:*

$$\text{Codim}_{X(w)}(X(w')) = \text{Card}(Q_w) - \text{Card}(Q_{w'}) = \Gamma(w', w) + q(w', w)$$

where

$$\Gamma(w', w) = \sum_{i=1}^s c_i(a_i + b_i) \quad \text{and} \quad q(w', w) = \frac{1}{2} \sum_{i=1}^{s+1} (c_i - c_{i-1})^2.$$

PROOF — This can be seen with a simple calculation. We will describe it geometrically on the quiver. We have the following quiver (see the appendix for a description of the quivers):

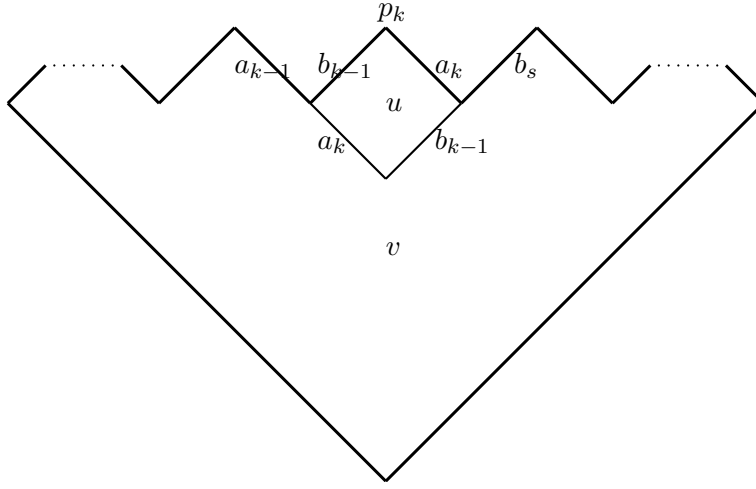


The codimension being given by the difference of the number of vertices, we find

$$\text{Codim}_{X(w)}(X(w')) = \sum_{i=1}^s c_i(a_i + b_i) - \sum_{i=1}^{s-1} c_i c_{i+1} + \sum_{i=1}^s c_i^2$$

and simple calculation gives the formula. Remark that we have $q(w', w) > 0$ for $w' \neq w$. \square

Let us assume that v is obtained from w by removing the k^{th} pic of Q_w . We obtain on the quivers the following situation:

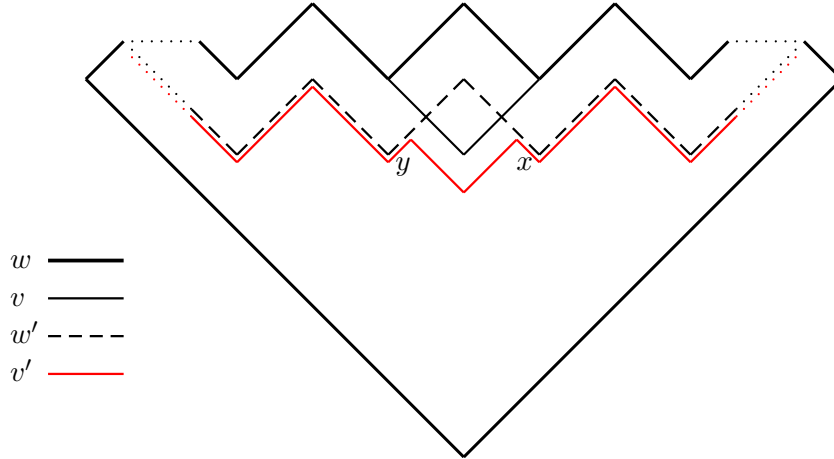


This simply means that the sequences of integers $(a_i(v))_{i \in [1, s]}$ and $(b_i(v))_{i \in [0, s-1]}$ are given by

$$a_i(v) = \begin{cases} a_i(w) & \text{for } i < k-1 \\ a_k(w) + a_{k-1}(w) & \text{for } i = k-1 \\ a_{i+1}(w) & \text{for } i \geq k \end{cases} \quad \text{and} \quad b_i(v) = \begin{cases} b_i(w) & \text{for } i < k-1 \\ b_k(w) + b_{k-1}(w) & \text{for } i = k-1 \\ b_{i+1}(w) & \text{for } i \geq k \end{cases}$$

where $a_0(w) = b_0(w)$. Furthermore, the quiver Q_u has no hole meaning that the variety \overline{PuQ}/Q is smooth and u' has to be equal to u . We only need to determine v' .

Let us now consider the quiver Q obtained by intersecting in Q_w the quivers Q_v and $Q_{w'}$. The quivers of v' has to be a subquiver of this quiver such that (see remark 7.6) all the holes of $Q_{w'}$ are holes of $Q_{v'}$ and $Q_{v'}$ may have one more hole corresponding to the hole of v which is not a hole of w .



The quiver $Q_{v'}$ has $s+1$ holes and the sequences $(a_i(v'))_{i \in [1, s+2]}$ and $(b_i(v'))_{i \in [0, s+1]}$ are given

by

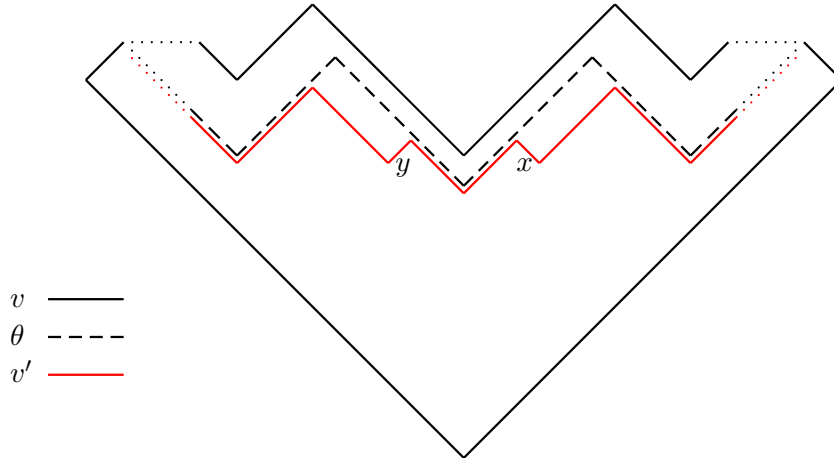
$$a_i(v') = \begin{cases} a_i(w') & \text{for } i \leq k-1 \\ a_k(w') - x & \text{for } i = k \\ x & \text{for } i = k+1 \\ a_{i-1}(w') & \text{for } i > k+1 \end{cases} \quad \text{and} \quad b_i(v') = \begin{cases} b_i(w') & \text{for } i < k-1 \\ y & \text{for } i = k-1 \\ b_{k-1}(w') - y & \text{for } i = k \\ b_{i-1}(w) & \text{for } i \geq k+1 \end{cases}$$

where $x \in [0, c_k]$, $y \in [0, c_{k-1}]$ and $c_{k-1} - y = c_k - x$. Indeed, the last formula is given by the fact that the only hole different from those of $Q_{w'}$ has to be associated to the same root as the hole of v which is not a hole of w . This gives the equality

$$\sum_{i=1}^{k-1} (a_i(v) + b_{i-1}(v)) = \sum_{i=1}^k (a_i(v') + b_{i-1}(v'))$$

and the equality $c_{k-1} - y = c_k - x$. The fact that $x \in [0, c_k]$ and $y \in [0, c_{k-1}]$ are equivalent to the fact that $X(v')$ is a Schubert subvariety of $X(v)$.

The Schubert subvariety $X(\theta)$ is contained in $X(v)$, contains $X(v')$ and is stable by the stabiliser of $X(v)$. It must have the same hole as v' except for those not corresponding to holes of v . In our case we have to fill the holes $k-1$ and $k+1$ of v' to obtain θ :



The quiver Q_θ of θ has $s-1$ holes and the integers $(a_i(\theta))_{i \in [1, s]}$ and $(b_i(\theta))_{i \in [0, s-1]}$ are given by:

$$a_i(\theta) = \begin{cases} a_i(v') & \text{for } i \leq k-2 \\ a_{k-1}(v') + a_k(v') & \text{for } i = k-1 \\ a_{k+1}(v') + a_{k+2}(v') & \text{for } i = k \\ a_{i+2}(v') & \text{for } i \geq k+1 \end{cases} \quad \text{and} \quad b_i(\theta) = \begin{cases} b_i(v') & \text{for } i < k-2 \\ b_{k-2}(v') + b_{k-1}(v') & \text{for } i = k-2 \\ b_k(v') + b_{k+1}(v') & \text{for } i = k-1 \\ b_{i+2}(w) & \text{for } i \geq k \end{cases}$$

It is now an easy calculation (and straightforward on the quiver) that the depth c'_i of θ in the holes of v is $c_{k-1} - y = c_k - x$ for the $(k-1)^{\text{th}}$, and c_i for all the holes before the $(k-1)^{\text{th}}$ hole and c_{i+1} for all the holes after the $(k-1)^{\text{th}}$ hole. We can now calculate:

$$\text{Codim}_{X(w)}(X(v)) - \text{Codim}_{X(w')}(X(v')) = a_k(w)b_{k-1}(w) - (a_k(w') - x)(b_{k-1}(w') - y)$$

and finally:

$$\text{Codim}_{X(w)}(X(v)) - \text{Codim}_{X(w')}(X(v')) = xa_k(w) + yb_{k-1}(w) - xy.$$

On the other hand we calculate

$$\begin{aligned} \Gamma(w', w) - \Gamma(\theta, v) &= \sum_{i=1}^s c_i(a_i(w) + b_i(w)) - \sum_{i=1}^{s-1} c'_i(a_i(v) + b_i(v)) \\ &= \sum_{i=1}^s c_i(a_i(w) + b_i(w)) - \sum_{i < k-1} c_i(a_i(w) + b_i(w)) - \sum_{i > k-1} c_{i+1}(a_{i+1}(w) + b_{i+1}(w)) - c'_{k-1}(a_{k-1}(v) + b_{k-1}(v)) \\ &= c_{k-1}(a_{k-1}(w) + b_{k-1}(w)) + c_k(a_k(w) + b_k(w)) - (c_k - x)(a_k(w) + b_k(w)) - (c_{k-1} - y)(a_{k-1}(w) + b_{k-1}(w)) \end{aligned}$$

and finally

$$\Gamma(w', w) - \Gamma(\theta, v) = x(a_k(w) + b_k(w)) + y(a_{k-1}(w) + b_{k-1}(w)).$$

Now the fact that the k^{th} pic of w was smaller than the adjacent pics means that we have $a_{k-1}(w) \geq b_{k-1}(w)$ and $b_k(w) \geq a_k(w)$ so that we get the inequality (because x and y are non negative):

$$2(\text{Codim}_{X(w)}(X(v)) - \text{Codim}_{X(w')}(X(v'))) \leq \Gamma(w', w) - \Gamma(\theta, v).$$

And the theorem is proved in this case.

7.3.2 The D_n case

Let us consider a minuscule quiver Q_w of a Schubert variety in the grassmannian $\mathbb{G}_{iso}(p, 2p)$ of p -dimensional isotropic subvector spaces of a $2p$ -dimensional vector space endowed with a non degenerate quadratic form. We may assume that all the simple roots are in the support of w (otherwise we simply restrict the group) so that there will be no virtual hole in Q_w . The set of simple roots $\beta(t(Q_w))$ can be written as $\{\alpha_{k_1}, \dots, \alpha_{k_s}\}$ in the notation of [Bou]. Let us denote by t_1, \dots, t_s the holes such that $\beta(t_i) = \alpha_{k_i}$. Because of proposition 4.2, for all $i \in [2, s]$ there exists exactly one pic p_i between the holes t_{i-1} and t_i . Furthermore there must be a pic p_1 with $\beta(p_1) = \alpha_k$ with $k < k_1$. If $\alpha_{k_s} \notin \{\alpha_{p-1}, \alpha_p\}$ then there must be pic p_{s+1} with $\beta(p_{s+1}) = \alpha_{p-1}$ or α_p and in this case there are $s+1$ pics, otherwise there are s pics. In the last case, we have $\alpha_{k_s} = \alpha_{p-1+i}$ with $i = 0$ or $i = 1$. We define p_{s+1} to be the smallest vertex (for \preceq) of Q_w with $\beta(p_{s+1}) = \alpha_{p-i}$.

Let us now define the following sequences $(a_i(w))_{i \in [1, s+1]}$ and $(b_i(w))_{i \in [0, s]}$ of integers (we will sometimes simply denote them by a_i and b_i omiting w):

$$\text{In the first case: } \begin{cases} a_i(w) = h(p_i) - h(t_i) \text{ for } i \in [1, s] \\ b_i(w) = h(p_{i+1}) - h(t_i) \text{ for } i \in [1, s-1], \\ b_s(w) = h(p_{s+1}) - h(t_s) + \frac{1}{2} \\ a_{s+1}(w) = p - \sum_{i=1}^s a_i(w), \\ b_0(w) = \frac{1}{2}h(p_{s+1}) - \sum_{i=1}^s b_i(w). \end{cases}$$

$$\text{In the second case: } \begin{cases} a_i(w) = h(p_i) - h(t_i) \text{ and } b_i(w) = h(p_{i+1}) - h(t_i) \text{ for } i \in [1, s-1], \\ a_s(w) = h(p_s) - h(p_{s+1}) \text{ and } b_s(w) = h(p_{s+1}) - h(p_{s+1}) - \frac{1}{2} \\ a_{s+1}(w) = p - \sum_{i=1}^s a_i(w), \\ b_0(w) = \frac{1}{2}h(p_{s+1}) - \sum_{i=1}^s b_i(w). \end{cases}$$

It is an easy game on the quiver and the description of configuration varieties to verify that if the associated sequences of integers to the quiver Q_w are $(a_i(w))_{i \in [1, s+1]}$ and $(b_i(w))_{i \in [0, s]}$ then we have:

$$X(w) = \{V \in \mathbb{G}_{iso}(p, 2p) / \dim(V \cap \mathbb{K}^{n_i}) \geq m_i \text{ for all } i \in [1, s]\}$$

where the \mathbb{K}^k form a complete flag of fixed isotropic subspaces, $n_i = \sum_{k=1}^i (a_k + b_{k-1})$ and $m_i = \sum_{k=1}^i a_k$ for $i \in [1, s]$ in the first case, and $n_i = \sum_{k=1}^i (a_k + b_{k-1})$ and $m_i = \sum_{k=1}^i a_k$ for $i \in [1, s-1]$, $n_s = p$, $m_s = 1 + \sum_{k=1}^s a_k$ and \mathbb{K}^{n_s} is associated to the root α_{k_s} in the second case.

Let $X(w')$ be a Schubert subvariety of $X(w)$ with the same stabiliser. Then we must have $\beta(t(Q_{w'})) \subset \beta(t(Q_w))$. For any hole t_i of Q_w let us define the depth of w' in t_i to be the integer

$$c_i = \begin{cases} \text{Card}\{j \in Q_w \setminus Q_{w'} / \beta(j) = \beta(t_i)\} \text{ for } \beta(t_i) \notin \{\alpha_{p-1}, \alpha_p\} \\ 2\text{Card}\{j \in Q_w \setminus Q_{w'} / \beta(j) = \beta(t_i)\} \text{ for } \beta(t_i) \in \{\alpha_{p-1}, \alpha_p\} \end{cases}.$$

The same game on the quiver and the description of configuration varieties shows that the associated sequences are given by

$$\begin{cases} a_i(w') = a_i(w) + c_i - c_{i-1} \text{ for all } i \in [1, s+1] \\ b_i(w') = b_i(w) + c_i - c_{i+1} \text{ for all } i \in [0, s] \end{cases}$$

with $c_0 = 0$ and $c_{s+1} = c_s$. We have

$$X(w) = \{V \in \mathbb{G}(p, q) / \dim(V \cap \mathbb{K}^{n_i}) \geq m_i + l_i \text{ for all } i \in [1, s]\}$$

with $l_i = \sum_{k=1}^i c_k$. These description enables us to give the following fact to calculate the codimension of $X(w')$ in $X(w)$ (we set $c_0 = c_{s+1} = 0$):

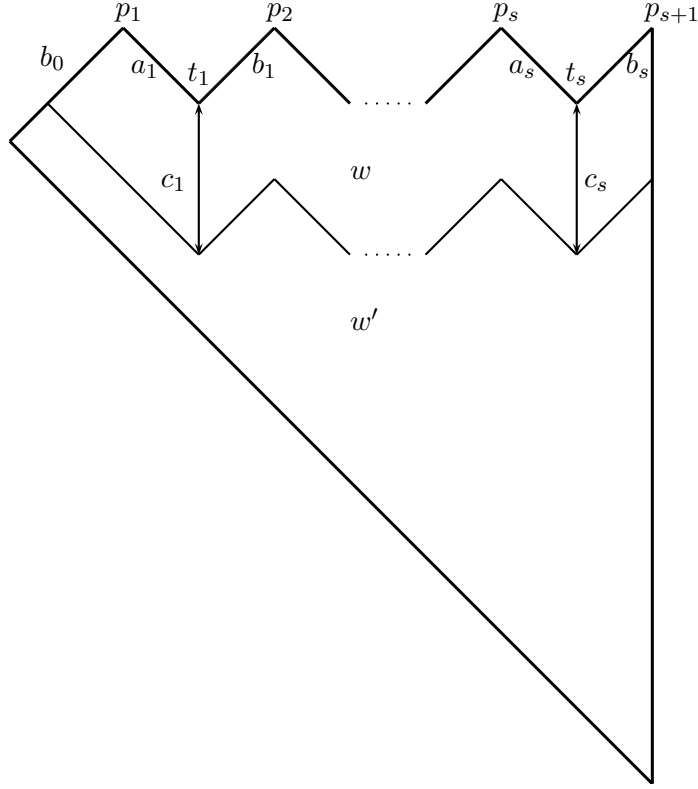
FACT 7.8. — *We have the formula:*

$$\text{Codim}_{X(w)}(X(w')) = \text{Card}(Q_w) - \text{Card}(Q_{w'}) = \Gamma(w', w) + q(w', w)$$

where

$$\Gamma(w', w) = \sum_{i=1}^s c_i(a_i + b_i) \quad \text{and} \quad q(w', w) = \frac{1}{2} \sum_{i=1}^s (c_i - c_{i-1})^2.$$

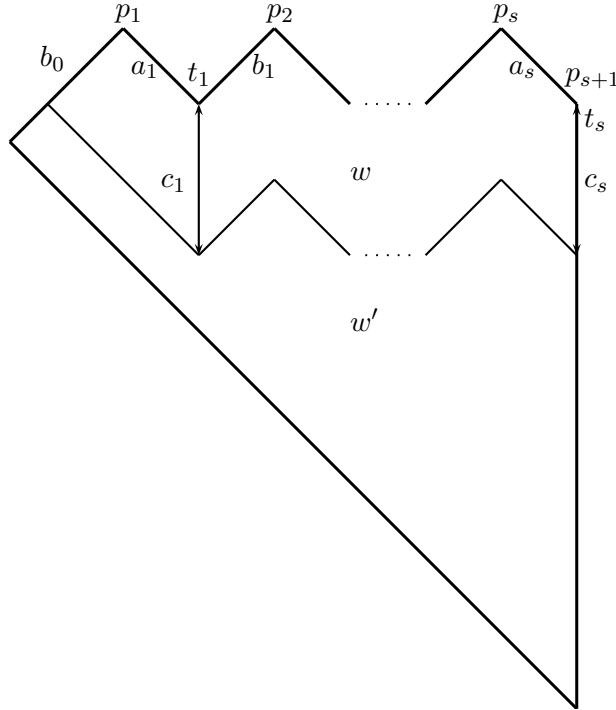
PROOF — This can be seen with a simple calculation. We will describe it geometrically on the quiver. In the first case, we have the following quiver:



The codimension being given by the difference of the number of vertices, we find

$$\text{Codim}_{X(w)}(X(w')) = \sum_{i=1}^{s-1} c_i(a_i + b_i) + c_s(a_s + b_s - \frac{1}{2}) - \frac{c_s(c_s - 1)}{2} - \sum_{i=1}^{s-1} c_i c_{i+1} + \sum_{i=1}^s c_i^2$$

and simple calculation gives the formula. In the second case, we have the following quiver:

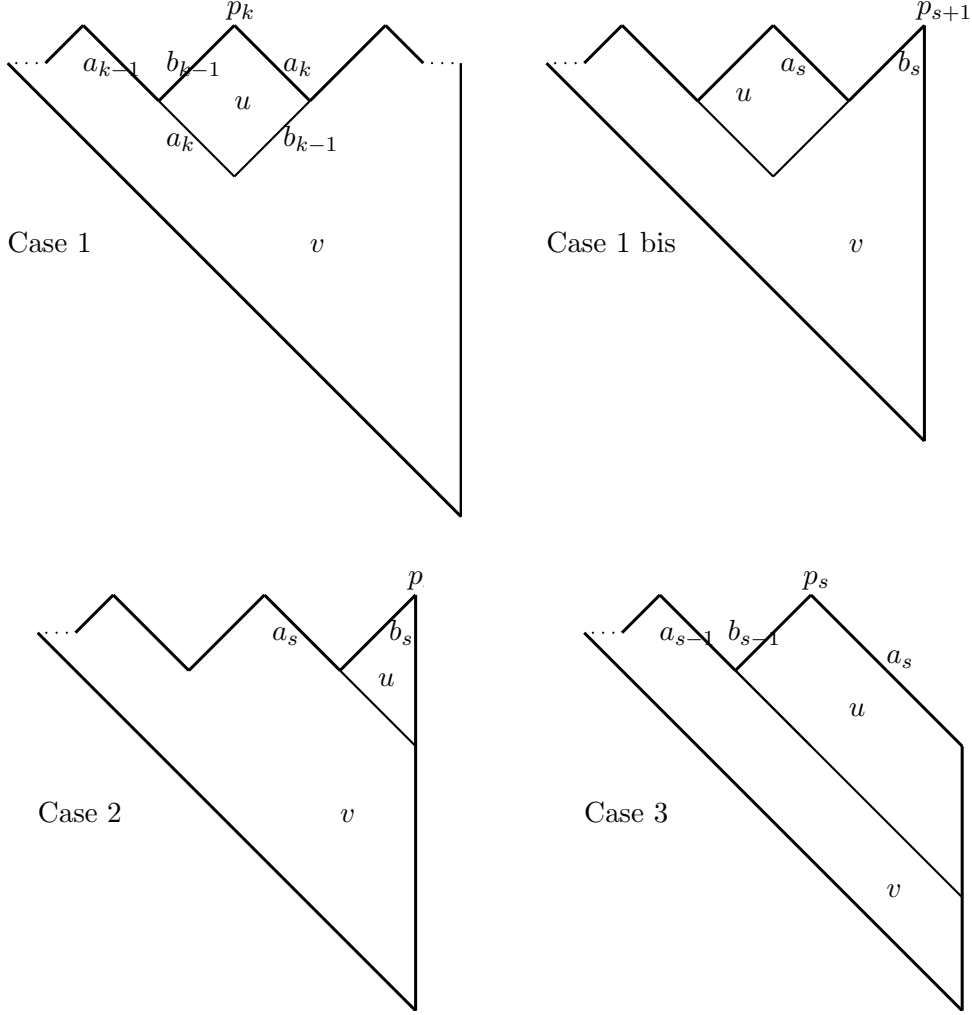


The codimension beeing given by the difference of the number of vertices, we find

$$\text{Codim}_{X(w)}(X(w')) = \sum_{i=1}^{s-1} c_i(a_i + b_i) + a_s c_s - \sum_{i=1}^{s-1} c_i c_{i+1} + \sum_{i=1}^{s-1} c_i^2 + \frac{c_s(c_s + 1)}{2}$$

and simple calculation gives the formula. Remark that we have $q(w', w) > 0$ for $w' \neq w$. \square

Let us assume that v is obtained from w by removing the k^{th} pic of Q_w . We obtain on the quivers the four following situation:



Case 1 is strictly equivalent to the A_n case and we get the result with the same calculation in this case. Case 1 bis has been done with these techniques in [SV1], we will not make the calculation one more time. For case 2, the sequences of integers $(a_i(v))_{i \in [1, s+1]}$ and $(b_i(v))_{i \in [0, s]}$ are given by:

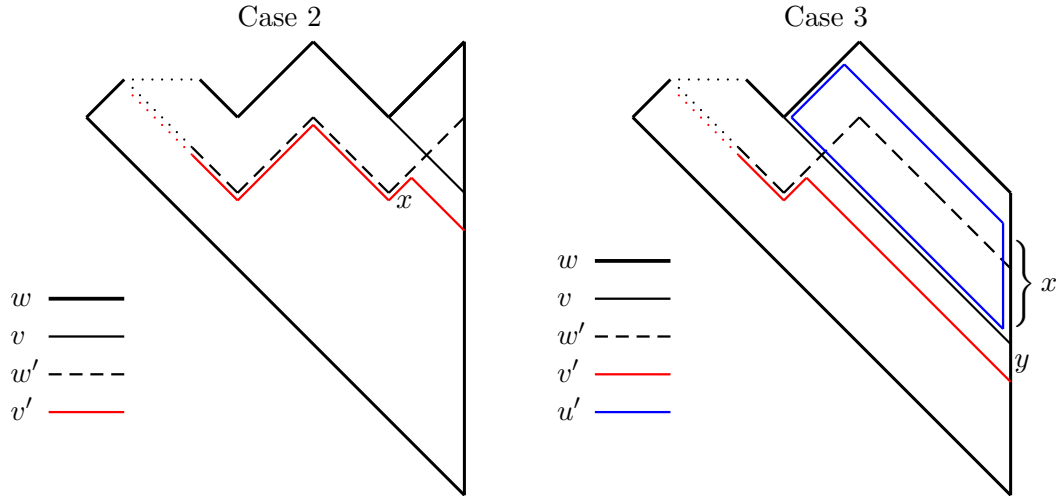
$$a_i(v) = \begin{cases} a_i(w) & \text{for } i < s \\ a_s(w) + b_s(w) - \frac{1}{2} & \text{for } i = s \\ a_{s+1}(w) - (b_s(w) - \frac{1}{2}) & \text{for } i = s + 1 \end{cases} \quad \text{and } b_i(v) = \begin{cases} b_0(w) + b_s(w) - \frac{1}{2} & \text{for } i = 0 \\ b_i(w) & \text{for } 0 < i < s \\ \frac{1}{2} & \text{for } i = s. \end{cases}$$

For case 3, the sequences of integers $(a_i(v))_{i \in [1, s]}$ and $(b_i(v))_{i \in [0, s-1]}$ are given by:

$$a_i(v) = \begin{cases} a_i(w) & \text{for } i < s-1 \\ a_s(w) + a_{s-1}(w) + b_{s-1}(w) & \text{for } i = s-1 \\ a_{s+1}(w) - b_{s-1}(w) & \text{for } i = s \end{cases} \quad \text{and} \quad b_i(v) = \begin{cases} b_i(w) & \text{for } i < s-1 \\ b_s(w) & \text{for } i = s-1. \end{cases}$$

Furthermore, in the case 2, the quiver Q_u has no hole meaning that the variety \overline{PuQ}/Q is smooth and u' has to be equal to u . In this case we only need to determine v' .

Let us now consider the quiver Q obtained by intersecting in Q_w the quivers Q_v and $Q_{w'}$. The quiver of v' has to be a subquiver of this quiver such that (see remark 7.6) all the holes of $Q_{w'}$ are holes of $Q_{v'}$ and $Q_{v'}$ may have one more hole corresponding to the hole of v which is not a hole of w .



In case 2, the quiver $Q_{v'}$ has $s+1$ holes and the sequences $(a_i(v'))_{i \in [1, s+2]}$ and $(b_i(v'))_{i \in [0, s+1]}$ are given by

$$a_i(v') = \begin{cases} a_i(w') & \text{for } i \leq s \\ b_s(w') - \frac{1}{2} - x & \text{for } i = s+1 \end{cases} \quad \text{and} \quad b_i(v') = \begin{cases} b_i(w') & \text{for } i < s \\ x & \text{for } i = s \\ \frac{1}{2} & \text{for } i = s+1 \end{cases}$$

where $x \in [0, c_s]$. In case 3, the quiver $Q_{v'}$ has s holes and the sequences $(a_i(v'))_{i \in [1, s+1]}$ and $(b_i(v'))_{i \in [0, s]}$ are given by

$$a_i(v') = \begin{cases} a_i(w') & \text{for } i \leq s-1 \\ a_s(w) + b_{s-1}(w) - c_{s-1} + y & \text{for } i = s \end{cases} \quad \text{and} \quad b_i(v') = \begin{cases} b_i(w') & \text{for } i < s-1 \\ c_{s-1} - y & \text{for } i = s-1 \\ b_s(w') & \text{for } i = s. \end{cases}$$

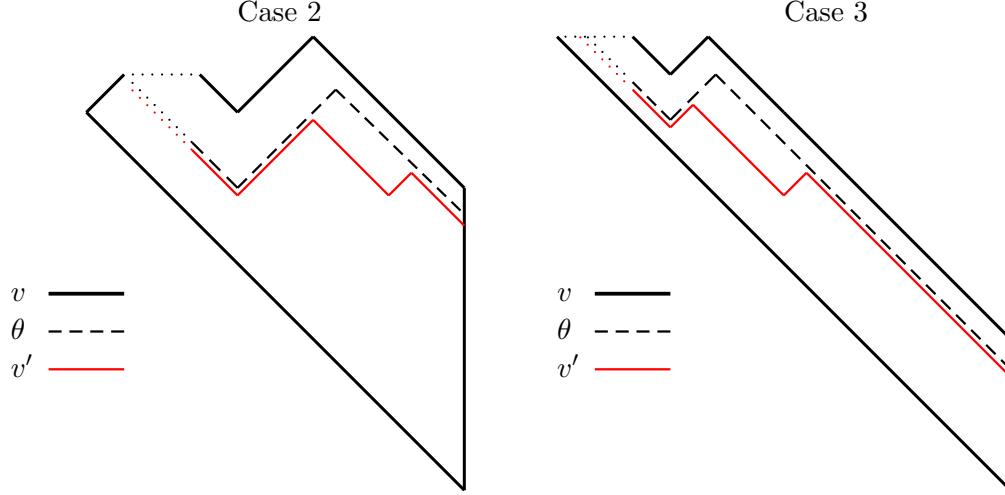
We also get that u' has a unique hole,

$$\begin{cases} a_0(u') = x \text{ and } a_1(u') = a_s(w) + b_{s-1}(w) - x = a_1(u) + b_0(u) - x \\ b_0(u') = x \text{ and } b_1(u') = b_1(u) = b_s(w) \end{cases}$$

In case 3, we have $y \in [c_{s-1} - c_s, c_{s-1}]$, $x \in [b_{s-1}(w') - c_k, b_{s-1}(w')]$ and $b_{s-1} - x = c_s - y$. Indeed, the last formula is given by the fact that last hole of $Q_{u'v'}$ has to be the same hole as

the last hole of $Q_{w'}$. This implies that $b_{s-1}(v') + b_0(u') = b_{s-1}(w')$ and the equality. The fact that $y \in [c_{s-1} - c_s, c_{s-1}]$ comes from the fact that $X(v')$ is a Schubert subvariety of $X(v)$ and that $X(u'v') = X(w')$.

The Schubert subvariety $X(\theta)$ is contained in $X(v)$, contains $X(v')$ and is stable by the stabiliser of $X(v)$. It must have the same hole as v' except for those not corresponding to holes of v . In our case we have to fill the s^{th} hole (in case 2) or the $(s-1)^{\text{th}}$ hole (in case 3) of v' to obtain θ :



In case 2, the integers $(a_i(\theta))_{i \in [1, s+1]}$ and $(b_i(\theta))_{i \in [0, s]}$ associated to θ are given by

$$a_i(\theta) = \begin{cases} a_i(v') & \text{for } i \neq s \\ a_{s+1}(v') + a_s(v') & \text{for } i = s \end{cases} \quad \text{and} \quad b_i(\theta) = \begin{cases} b_i(v') & \text{for } i \neq s-1 \\ b_{s-1}(v') + b_s(v') & \text{for } i = s-1 \end{cases}$$

In case 3, the integers associated to θ are given by $(a_i(\theta))_{i \in [1, s]}$ and $(b_i(\theta))_{i \in [0, s-1]}$ are given by

$$a_i(\theta) = \begin{cases} a_i(v') & \text{for } i \neq s-1 \\ a_{s-1}(v') + a_s(v') & \text{for } i = s-1 \end{cases} \quad \text{and} \quad b_i(\theta) = \begin{cases} b_i(v') & \text{for } i \neq s-2 \\ b_{s-2}(v') + b_{s-1}(v') & \text{for } i = s-2 \end{cases}$$

It is now an easy calculation (and straightforward on the quiver) that in case 2, the depth c'_i of θ in the holes of v is $c_s - x$ for the s^{th} hole and c_i for the other holes. In case 3, the depth c'_i of θ in the holes of v is $y = c_s + x - b_{s-1}$ for the $(s-1)^{\text{th}}$ hole and c_i for the other holes.

We can now calculate in case 2:

$$\dim X(u') - \text{Codim}_{X(w')}(X(v')) = \frac{(b_s(w) + \frac{1}{2})(b_s(w) - \frac{1}{2})}{2} - \frac{(b_s(w') - x + \frac{1}{2})(b_s(w') - x - \frac{1}{2})}{2}$$

and because $b_s(w') = b_s(w)$ we get:

$$\dim X(u') - \text{Codim}_{X(w')}(X(v')) = xb_s(w) - \frac{x^2}{2}.$$

On the other hand we have,

$$\Gamma(w', w) - \Gamma(\theta, v) = \sum_{i=1}^s c_i(a_i(w) + b_i(w)) - \sum_{i=1}^s c'_i(a_i(v) + b_i(v)),$$

a simple calculation gives

$$\Gamma(w', w) - \Gamma(\theta, v) = x(a_s(w) + b_s(w)).$$

Now the fact that the s^{th} pic of w was smaller than the $(s-1)^{\text{th}}$ pic means that $a_s(w) \geq b_s(w)$ so that we get the inequality :

$$2(\dim X(u') - \text{Codim}_{X(w')}(X(v'))) \leq \Gamma(w', w) - \Gamma(\theta, v).$$

The theorem is proved in case 2. In case 3, we have:

$$\begin{aligned} \dim X(u') - \text{Codim}_{X(w')}(X(v')) &= \frac{x(x+1)}{2} + x(a_s(w) + b_{s-1}(w) - x) - \frac{x(x+1)}{2} - xa_s(w') \\ &= x(a_s(w) + b_{s-1}(w) - x - a_s(w) - c_s + c_{s-1}) \end{aligned}$$

and finally:

$$\dim X(u') - \text{Codim}_{X(w')}(X(v')) = x(c_{s-1} - y).$$

On the other hand we have,

$$\Gamma(w', w) - \Gamma(\theta, v) = \sum_{i=1}^s c_i(a_i(w) + b_i(w)) - \sum_{i=1}^{s-1} c'_i(a_i(v) + b_i(v))$$

a simple calculation gives

$$\Gamma(w', w) - \Gamma(\theta, v) = (c_{s-1} - y)(a_{s-1}(w) + b_{s-1}(w)) + (b_{s-1} - x)(a_s(w) + b_s(w)).$$

Now the fact that the $(s-1)^{\text{th}}$ pic of w is smaller than the $(s-2)^{\text{th}}$ pic means that we have $a_{s-1}(w) \geq b_{s-1}(w)$. Furthermore, we have $x \leq b_{s-1}(w') = b_{s-1}(w)$ and $a_s(w) + b_s(w) \geq 0$ so that we get the inequality :

$$2(\dim X(u') - \text{Codim}_{X(w')}(X(v'))) \leq \Gamma(w', w) - \Gamma(\theta, v).$$

The theorem is proved in case 3.

7.4 Exceptional cases

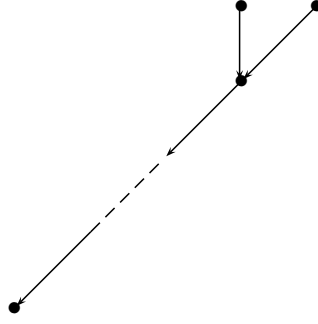
We are left to deal with three cases: quadrics and minuscule varieties for E_6 and E_7 .

7.4.1 Quadrics

For quadrics, let us remark that all Schubert varieties except one are locally factorial (the quivers have only one pic) so that in all cases except one we have $\widehat{X}(\widehat{w}) = X(w)$ and there is nothing to prove. The only non locally factorial Schubert variety (we are in \mathbb{K}^{2p} with a non degenerate quadratic form) is given by:

$$X(w) = \{x \in \mathbb{P}(\mathbb{K}^{2p}) / x \text{ is isotropic and } x \in W_{p-2}^\perp\}$$

for a fixed isotropic subspace W_{p-2} of dimension $p-2$. The associated quiver has p vertices and is given by



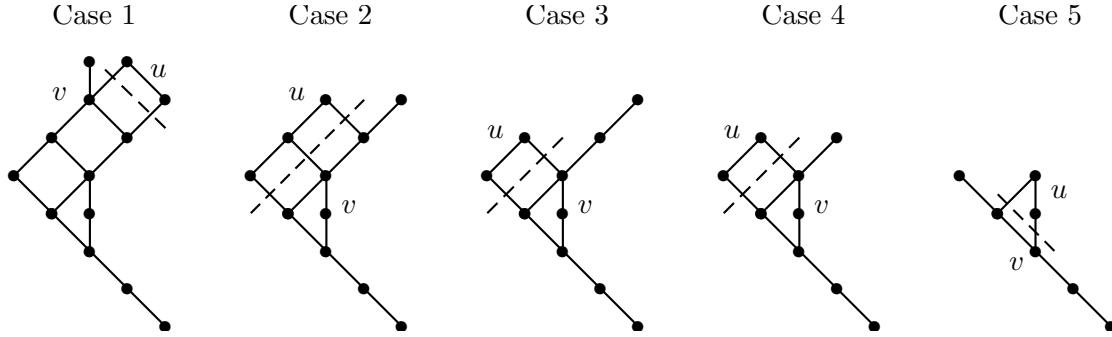
In particular, the resolution $\hat{\pi} : \hat{X}(\hat{w}) \rightarrow X(w)$ is given by $p : \overline{PuQ} \times^Q X(v) \rightarrow X(w)$ where \overline{PuQ}/Q is of dimension 1. The fiber of the morphism $\hat{\pi}$ is at most 1. On the other hand, as it is $\text{Stab}(X(w))$ -equivariant, the fiber is strictly positive on Schubert subvarieties stable under $\text{Stab}(X(w))$. These subvarieties are of codimension at least 3 and the result follows.

More generally, if the morphism $\hat{\pi}$ is of the form $p : \overline{PuQ} \times^Q X(v) \rightarrow X(w)$ and its fiber is of dimension at most one then the morphism $\hat{\pi}$ has to be *IH*-small.

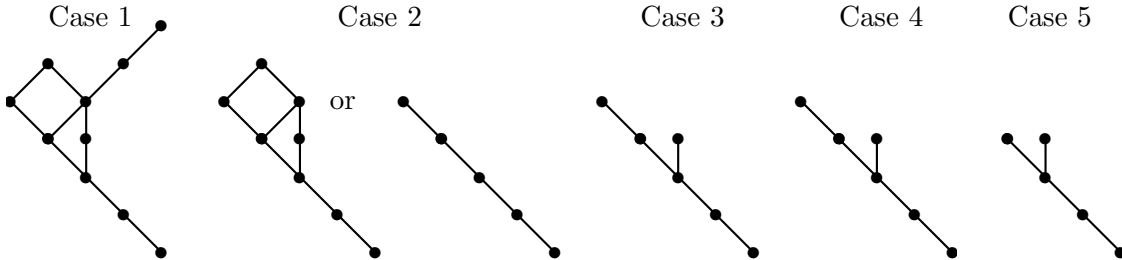
7.4.2 The E_6 case

We have seen that if the Schubert variety is locally factorial (i.e. its quiver has a unique pic) or if the morphism $\hat{\pi}$ is of the form $p : \overline{PuQ} \times^Q X(v) \rightarrow X(w)$ and its fiber is of dimension at most one then the morphism $\hat{\pi}$ has to be *IH*-small. We are now going to list the morphisms $\hat{\pi}$ obtained from construction 3 not satisfying these properties and verify that $\hat{\pi}$ is small.

For E_6 , all the morphisms $\hat{\pi}$ not verifying the preceding properties are of the form $p : \overline{PuQ} \times^Q X(v) \rightarrow X(w)$. We list here the quiver of $X(w)$ indicating on each quiver the quivers of u and v .



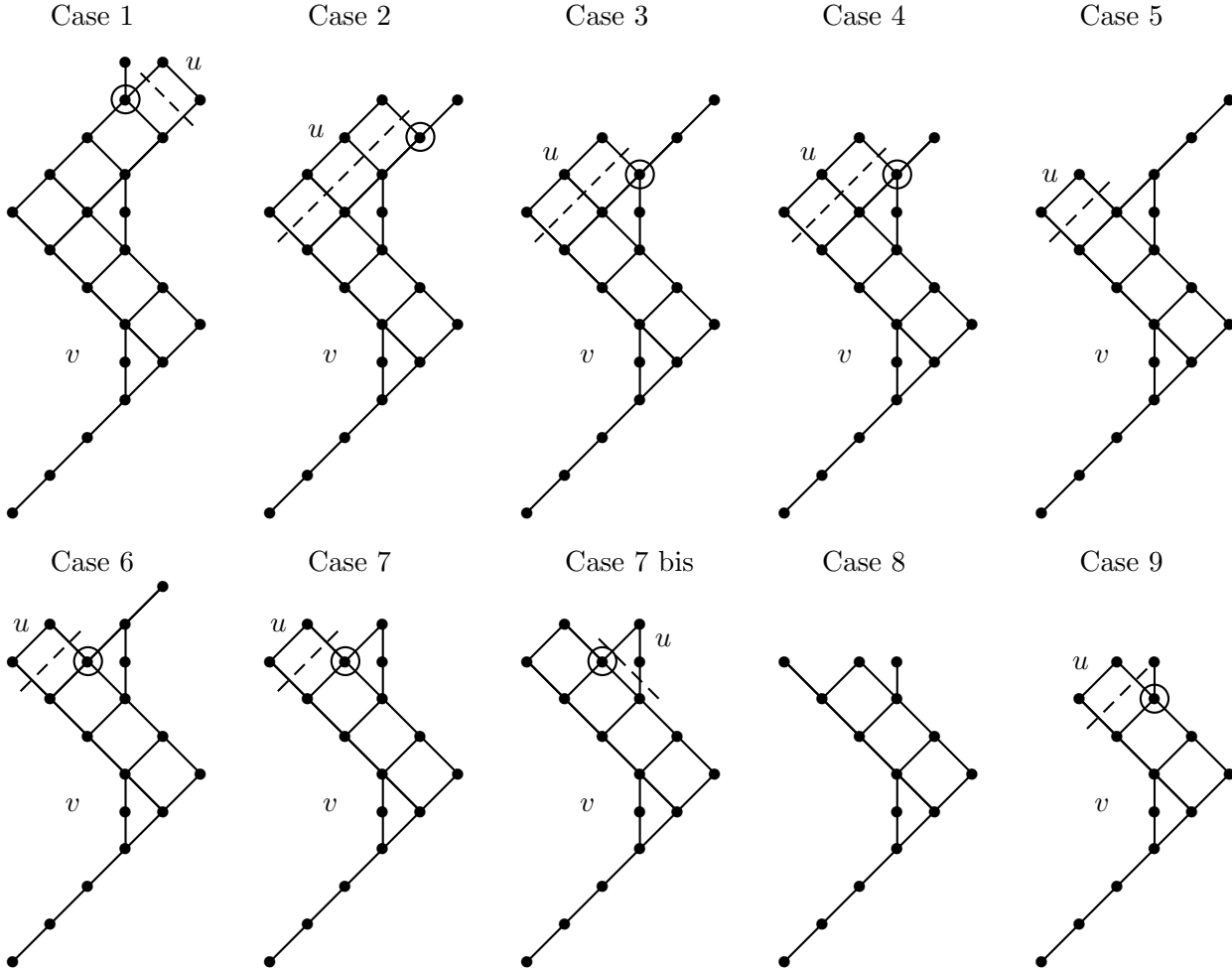
The dimension of the fiber in these morphisms is at most $f = 2$ except in the second case where it is at most $f = 3$. The Schubert subvarieties $X(w')$ stable under $\text{Stab}(X(w))$ of codimension not bigger than $2f$ are the following:

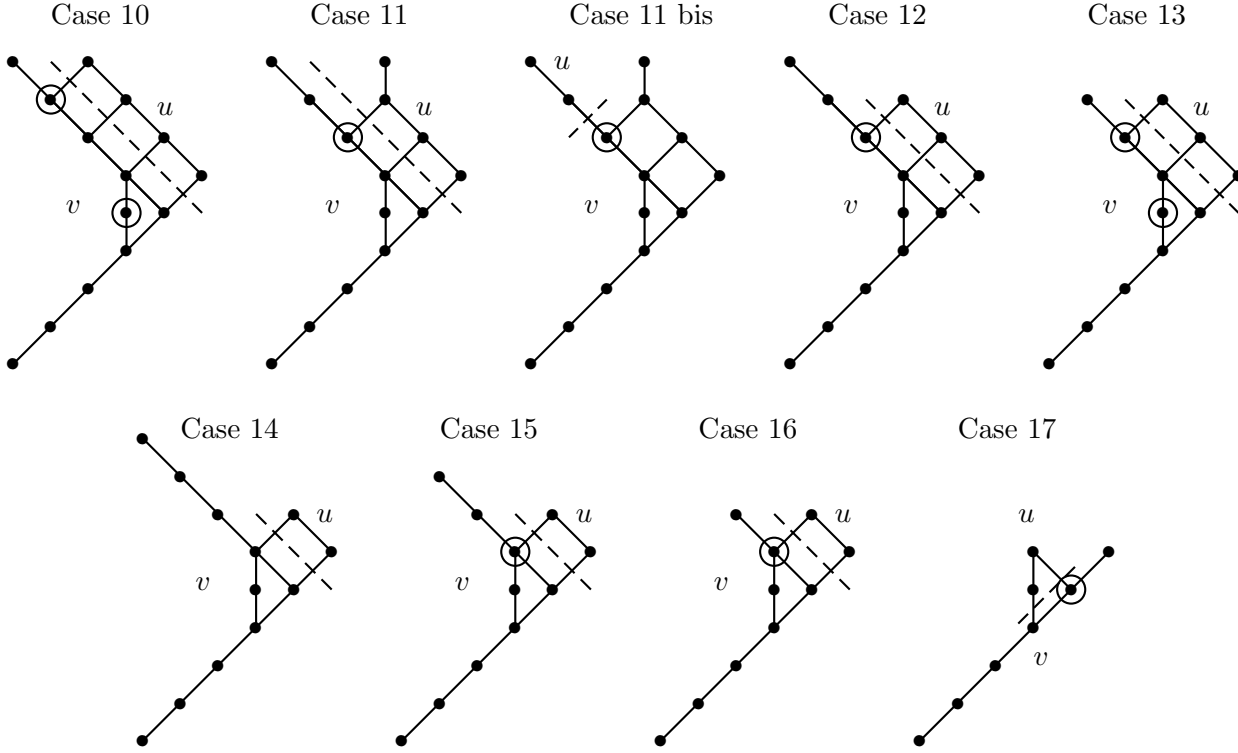


These quivers Q are obtained from the quiver Q_w by removing all the vertices smaller than a hole i of Q_w . It is now easy to see that any subvariety Z_K of the Bott-Samelson resolution $\tilde{\pi} : \tilde{X}(\tilde{w}) \rightarrow X(w)$ such that $\tilde{\pi}(Z_K) = X(w')$ is contained in the divisor Z_i . We thus have $\tilde{\pi}^{-1}(X(w')) = Z_i$ and $\hat{\pi}^{-1}(X(w'))$ is contained in the image of Z_i in $\hat{X}(\hat{w})$. Seeing $\hat{X}(\hat{w})$ as a configuration variety, the image of Z_i in $\hat{X}(\hat{w})$ is the configuration variety $\overline{PuQ} \times^Q X(v')$ where $Q_{v'} = Q \cap Q_v$. In particular, the dimension of the fiber of $\hat{\pi}$ above $X(w')$ is 1, 1, 2, 1, 1, 1 in the different cases and the morphism $\hat{\pi}$ is always IH -small.

7.4.3 The E_7 case

We proceed in the same way in this case and list the quivers having at least two pics and for which the fiber is at least two.





All the resolutions are of type $\overline{PuQ} \times^Q X(v')$ except case 8. The maximal dimension f of the fiber in all these cases is given by

case	1	2	3	4	5	6	7	7 bis	9	10	11	11 bis	12	13	14	15	16	17
f	2	4	3	3	2	2	2	2	2	4	4	2	3	3	2	2	2	2

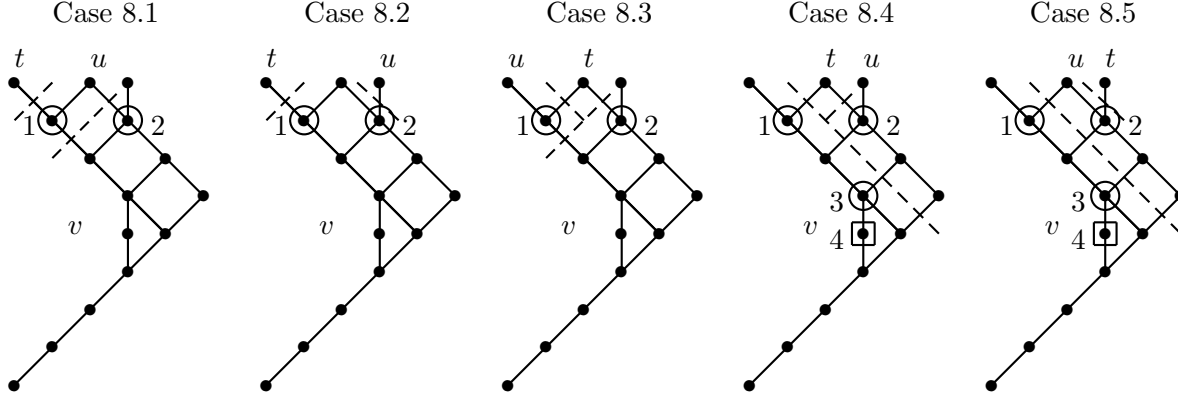
We have cerced the vertices i such that the quivers $Q_{w'}$ obtained from the quiver Q_w by removing all the vertices smaller (for \preccurlyeq) than the hole i of Q_w are the quivers of the Schubert subvarieties $X(w')$ stable under $\text{Stab}(X(w))$ of codimension not superior to $2f$. The codimension of $X(w')$ in $X(w)$ is given by:

case	1	2	3	4	6	7	7 bis	9	10	11	11 bis	12	13	15	16	17
codim	3	3	4	3	4	3	3	3	3 or 8	5	5	4	3 or 6	4	3	3

Remark that in case 5 and 14 there is no such Schubert subvariety so that the morphism is already small. In all the other cases and as for the E_6 case, it is easy to see that any subvariety Z_K of the Bott-Samelson resolution $\tilde{\pi} : \tilde{X}(\tilde{w}) \rightarrow X(w)$ such that $\tilde{\pi}(Z_K) = X(w')$ is contained in the divisor Z_i . We thus have $\tilde{\pi}^{-1}(X(w')) = Z_i$ and $\hat{\pi}^{-1}(X(w'))$ is contained in the image of Z_i in $\hat{X}(\hat{w})$. Seeing $\hat{X}(\hat{w})$ as a configuration variety, the image of Z_i in $\hat{X}(\hat{w})$ is the configuration variety $\overline{PuQ} \times^Q X(v')$ where $Q_{v'} = Q_{w'} \cap Q_v$. In particular, the dimension of the fiber of $\hat{\pi}$ above $X(w')$ is given by

case	1	2	3	4	6	7	7 bis	9	10	11	11 bis	12	13	15	16	17
dim	1	1	1	1	1	1	1	1	1 or 3	2	2	1	1 or 2	1	1	1

and the morphism $\hat{\pi}$ is always IH -small in these cases. We are left with case 8 for which the resolution is of the form $\overline{PtQ} \times^Q \overline{RuS} \times^S X(v)$ and the partitions of the quivers are given by:



The maximal dimension f of the fiber in all these cases is given by

case	8.1	8.2	8.3	8.4	8.5
f	3	2	3	5	5

We have circled and numeroted the vertices i such that the quivers $Q_{w'}$ obtained from the quiver Q_w by removing all the vertices smaller (for \preccurlyeq) than a fixed subset of the holes of Q_w are the quivers of the Schubert subvarieties $X(w')$ stable under $\text{Stab}(X(w))$ of codimension not superior to $2f$. Let A be a non empty subset of $\{1, 2, 3\}$ and let $Q_{w'}$ be the quiver obtained by removing the vertices smaller than the vertices in A . The codimension of $X(w')$ in $X(w)$ is given by (here A is $\{1\}$, $\{2\}$, $\{1, 2\}$ or in the last two cases $\{3\}$):

case	8.1	8.2	8.3	8.4	8.5
codim	3, 3 or 5	3, 3 or 5	3, 3 or 5	3, 3, 5 or 8	3, 3, 5 or 8

Suppose that A is a subset of $\{1, 2\}$, it is easy to see that any subvariety Z_K of the Bott-Samelson resolution $\tilde{\pi} : \tilde{X}(\tilde{w}) \rightarrow X(w)$ such that $\tilde{\pi}(Z_K) = X(w')$ is contained in the variety Z_A . The fiber $\hat{\pi}^{-1}(X(w'))$ is thus contained in the image of Z_A in $\hat{X}(\hat{w})$. Seeing $\hat{X}(\hat{w})$ as a configuration variety, this image in $\hat{X}(\hat{w})$ is the configuration variety $\overline{PtQ} \times^Q \overline{Ru'S} \times^S X(v')$ where $Q_{u'}$ and $Q_{v'}$ are obtained respectively from Q_u and Q_v by removing the vertices smaller than one vertex in $A \cap Q_u$ respectively $A \cap Q_v$. In particular, the dimension of the fiber of $\hat{\pi}$ above $X(w')$ is given by

case	8.1	8.2	8.3	8.4	8.5
dim	1, 1 or 1	1, 1 or 2	1, 1 or 1	1, 1 or 1	1, 1 or 1

and the morphism $\hat{\pi}$ is always IH -small in these cases. We are left with the case where $A = \{3\}$. In this case it is not hard to see that any subvariety Z_K of the Bott-Samelson resolution $\tilde{\pi} : \tilde{X}(\tilde{w}) \rightarrow X(w)$ such that $\tilde{\pi}(Z_K) = X(w')$ is contained in the variety $Z_{\{2,3\}}$ or in Z_4 . The fiber $\hat{\pi}^{-1}(X(w'))$ is thus contained in the image of $Z_{\{2,3\}}$ or of Z_4 in $\hat{X}(\hat{w})$. Seeing $\hat{X}(\hat{w})$ as a configuration variety, the image of $Z_{\{2,3\}}$ in $\hat{X}(\hat{w})$ is the configuration variety $\overline{PtQ} \times^Q \overline{Ru'S} \times^S X(v')$ where $Q_{u'}$ and $Q_{v'}$ are obtained respectively from Q_u and Q_v by removing the vertices smaller than one vertex in $\{2, 3\} \cap Q_u$ respectively $\{2, 3\} \cap Q_v$. The image of Z_4 is the configuration variety $\overline{PtQ} \times^Q \overline{Ru'S} \times^S X(v')$ where $Q_{v'}$ is obtained from Q_v by removing the

vertices smaller than the vertex 4. In particular, the fiber of $\hat{\pi}$ above $X(w')$ has two components whose dimensions are given by

case	8.4 ; $Z_{\{2,3\}}$	8.4 ; Z_4	8.5 ; $Z_{\{2,3\}}$	8.5 ; Z_4
dim	2	3	2	3

and the morphism $\hat{\pi}$ is always IH -small.

7.5 Small resolution

Let us now describe all IH -small resolutions of minuscule Schubert varieties whenever they exist. Having adopted (and described) the relative minimal models point of view, we use the following result of B. Totaro [To] using a key result of J. Wisniewski [Wi]:

THEOREM 7.9. — *Any IH -small resolution of X is a small relative minimal model for X .*

Looking for IH -small resolution we only have to check in our list of minimal models. Furthermore, because of theorem 7.3, the morphism $\hat{\pi} : \hat{X}(\hat{w}) \rightarrow X(w)$ from any minimal model to $X(w)$ is IH -small so that we get the following

COROLLARY 7.10. — *The IH -small resolutions of $X(w)$ are given by the morphisms $\hat{\pi} : \hat{X}(\hat{w}) \rightarrow X(w)$ obtained from construction 3 with $\hat{X}(\hat{w})$ smooth.*

We now give a combinatorial description of these varieties. Let Q_v be a quiver associated to a minuscule Schubert variety $X(v)$ and i a vertex of Q_v .

DEFINITION 7.11. — *The vertex i of Q_v is called minuscule if $\beta(i)$ is a minuscule simple root of the sub-Dynkin diagram of G defined by $\text{Supp}(v)$.*

Construction 3 gives a partition of the quiver Q_w into subquivers Q_{w_i} which are quivers of minuscule Schubert varieties having only one pic. We have

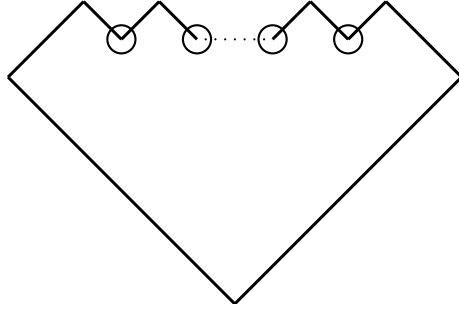
THEOREM 7.12. — *The variety $\hat{X}(\hat{w})$ obtained from construction 1 is smooth if and only if for all i , the unique pic p_i of Q_{w_i} is minuscule in Q_{w_i} .*

PROOF — We have seen that the variety $\hat{X}(\hat{w})$ is a sequence of locally trivial fibrations with fiber Schubert varieties $X(w_i)$. The theorem will follow from: \square

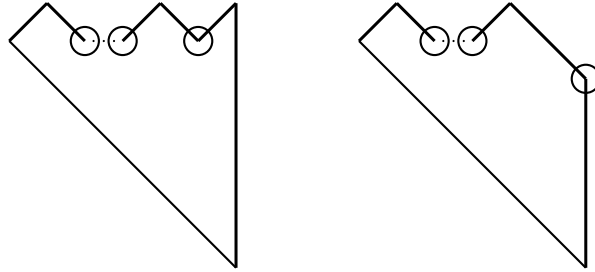
PROPOSITION 7.13. — *A minuscule Schubert variety $X(w)$ is smooth if and only if Q_w has a unique pic p and p is minuscule in Q_w .*

PROOF — We know from [BP] that a minuscule Schubert variety $X(w)$ is smooth if and only if it is homogeneous under its stabiliser. It is easy to verify that the quiver of any minuscule homogeneous variety has a unique pic which is minuscule.

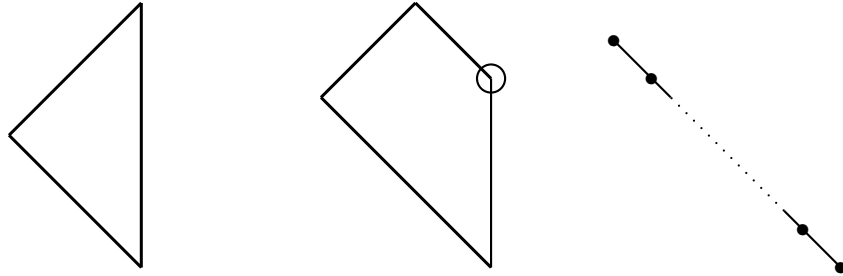
Conversely, according to proposition 4.11, the variety is homogeneous under its stabiliser if and only if the quiver Q_w has no non virtual hole. Now we have seen that for A_n the quiver of any Schubert variety is of the form (we have cerced the non virtual holes of the quiver):



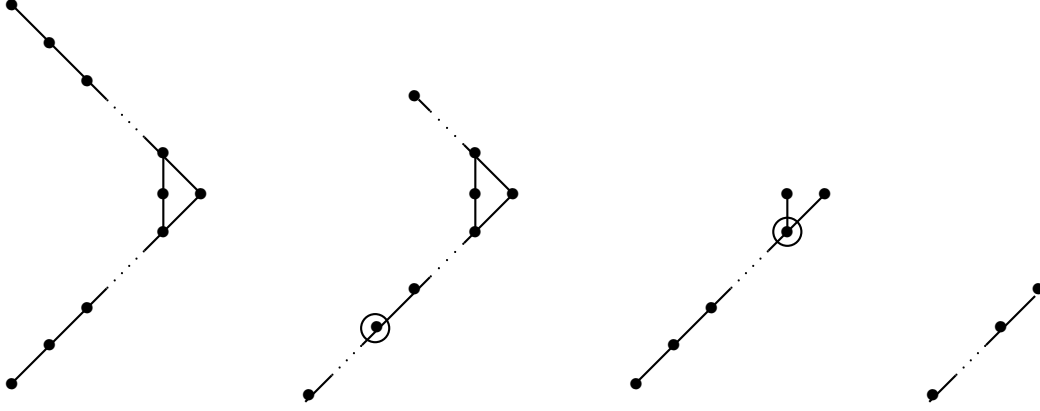
and the only case where there is a unique pic is when there is no hole. In this case we have the quiver of a grassmannian and it is smooth see appendix). For the case of maximal isotropic subspaces (say associated to the simple root α_n with the notations of [Bou]), the quiver is of the form (we have cerced the non virtual holes of the quiver):



and there are three cases when there is only one pic namely



In the second case, one of the two vertices i_{n-1} and i_n such that i_k is the smallest element (for \preccurlyeq) with $\beta(i_k) = \alpha_k$ with the notations of [Bou] is a hole of the quiver. In the first case the quiver is the quiver of the isotropic grassmannian and in the third one it is the quiver of a projective space. For the quadric case, the quiver has one of the four following forms:



and in the first and last cases we get respectively the quiver of a quadric or the quiver of a projective space. In the two intermediate cases, there is one hole in the quiver.

Finally, it is an easy verification on the quivers of E_6 and E_7 to check that the proposition is true (cf. appendix). \square

7.6 Stringy polynomials

Another way of proving the non existence of IH -small resolutions is the following: because of theorem 7.9 of any IH -small resolution \tilde{X} of a variety X if it exists will factor through the relative canonical model \hat{X} (which will always exists if \tilde{X} does). Furthermore, the resolution $\tilde{X} \rightarrow \hat{X}$ will be IH -small and in particular crepant. We can thus use the stringy polynomial $E(\hat{X}, u, v)$ defined by V. Batyrev in [Ba]. If \hat{X} admits a crepant resolution then this polynomial (which in general is a formal power serie) is a true polynomial. To prove the non existence of IH -resolution, it would be enough to prove that $E(\hat{X}, u, v)$ is not a polynomial.

Let us give an example where we make the full calculation. Let us first recall the following definitions (for more details and more general definitions, see [Ba]).

Let X be a normal irreducible variety, we define the following notations:

$$E(X, u, v) = \sum_{u, v} e^{p, q}(X) u^p v^q \quad \text{with} \quad e^{p, q}(X) = \sum_i (-1)^i h^{p, q}(H_c^i(X, \mathbb{C}))$$

where $H_c^i(X, \mathbb{C})$ is the i^{th} cohomology group with compact support and $h^{p, q}(H_c^i(X, \mathbb{C}))$ is the dimension of its (p, q) -type component. The polynomial $E(X, u, v)$ is what V. Batyrev call the Euler polynomial (or E -polynomial).

Assume now that X is a gorenstein normal irreducible variety with at worst terminal singularities. Let $\pi : Y \rightarrow X$ a resolution of singularities such that the exceptional locus is a divisor D whose irreducible components $(D_i)_{i \in I}$ are smooth divisors with only normal crossing. We then have

$$K_Y = \pi^* K_X + \sum_{i \in I} a_i D_i \quad \text{with} \quad a_i > 0.$$

For any subset $J \subset I$ we define

$$D_J = \begin{cases} \bigcap_{j \in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases} \quad \text{and} \quad D_J^o = D_J \setminus \bigcup_{i \in I \setminus J} (D_J \cap D_i).$$

DEFINITION 7.14. — *The stringy function associated with the resolution $\pi : Y \rightarrow X$ is the following:*

$$E_{\text{st}}(X, u, v) = \sum_{J \subset I} E(D_J^o, u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}.$$

Then V. Batyrev proves the following

THEOREM 7.15. — *(i) The function $E_{\text{st}}(X, u, v)$ is independent of the resolution $\pi : Y \rightarrow X$ with exceptional locus of pure codimension 1 given by smooth irreducible divisors with normal crossing.*

(ii) If X admits a crepant resolution $\pi : Y \rightarrow X$ (that is to say $\pi^ K_X = K_Y$) then $E_{\text{st}}(X, u, v) = E(Y, u, v)$ and it is a polynomial.*

(iii) In particular, if X admits a crepant resolution, the stingy Euler number

$$e_{\text{st}}(X) = \lim_{u, v \rightarrow 0} E_{\text{st}}(X, u, v) = \sum_{J \subset I} e(D_J^o) \prod_{j \in J} \frac{1}{1 + a_j}$$

is an integer.

We now give an example of a minuscule Schubert variety which is singular non locally factorial and does not admit an IH -small resolution.

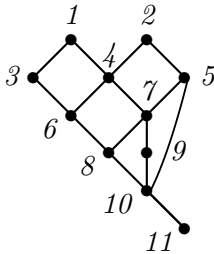
EXAMPLE 7.16. — *Let G be $SO(12)$ and w given by the following reduced writing \tilde{w} (the symmetry s_i is the simple reflection associated to the i^{th} simple root with the notation of [Bou]):*

$$w = s_2 s_4 s_1 s_3 s_6 s_2 s_4 s_3 s_4 s_6.$$

The associated Schubert variety is the following ($\mathbb{G}_{iso}(k, 12)$ is the isotropic grassmannian, we denote by $\mathbb{G}_{iso}^1(6, 12)$ and $\mathbb{G}_{iso}^2(6, 12)$ the homogenous varieties associated to the simple roots α_5 and α_6):

$$X(w) = \{V \in \mathbb{G}_{iso}^2(6, 12) \mid \dim(V \cap W_3) \geq 1 \text{ and } \dim(V \cap W_6) \geq 3\}$$

where $W_3 \in \mathbb{G}_{iso}(3, 12)$ and $W_6 \in \mathbb{G}_{iso}^1(6, 12)$. The quiver Q_w is



The variety $X(w)$ has for resolution the Bott-Samelson resolution $\tilde{X}(\tilde{w})$. Moreover, because the morphism $\pi : \tilde{X}(\tilde{w}) \rightarrow X(w)$ is B -equivariant, the exceptional locus has to be B -invariant and thus an union of Z_K . The only non contracted divisors Z_i of $\tilde{X}(\tilde{w})$ in $X(w)$ are Z_1 and Z_2 . Furthermore, the variety $Z_{\{1,2\}}$ is not contracted so that the exceptional locus D is the union

$$D = \cup_{i=3}^{11} Z_i.$$

All Z_i are smooth and intersect transversally. Denote by D_1 and D_2 the images of Z_1 and Z_2 in $X(w)$. The ample generator of the Picard group of $X(w)$ is given by $L = D_1 + D_2$. We have

$$\pi^* L = \sum_{i=1}^{11} Z_i.$$

Formulae of paragraph 3.3.1 and lemma 4.16 give us:

$$-K_{\tilde{X}(\tilde{w})} = \sum_{i=1}^{11} (h(i) + 1) Z_i$$

and proposition 4.15 gives us:

$$-K_{X(w)} = 7D_1 + 7D_2 = 7L.$$

In particular, we have

$$K_{\tilde{X}(w)} - \pi^* K_{X(w)} = \sum_{i=1}^{11} (6 - h(i)) Z_i = (Z_3 + Z_4 + Z_5) + 2(Z_6 + Z_7) + 3(Z_8 + Z_9) + 4Z_{10} + 5Z_{11}.$$

Remark that for $J \subset [3, 9]$, the variety Z_J^o is a sequence of 9 locally trivial fibrations in \mathbb{A}^1 of in points (there are exactly $|J|$ points) over $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, we have $e(Z_J^o) = 4$ for all $J \subset [3, 9]$.

Now we have the easy formula

$$\sum_{J \subset I} \prod_{j \in J} x_j = \prod_{i \in I} (1 + x_i).$$

We can thus calculate in our situation:

$$e_{\text{st}}(X(w)) = 4(1 + \frac{1}{2})^3(1 + \frac{1}{3})^2(1 + \frac{1}{4})^2(1 + \frac{1}{5})(1 + \frac{1}{6}) = \frac{105}{2}.$$

We conclude that $X(w)$ has no IH -small resolution as given by theorem 7.12.

This kind of calculation can be generalised, this will be done in a subsequent paper. For example, the same calculation in the general case where $X(w)$ is gorenstein (or equivalently all the pics $p \in p(Q_w)$ have the same height $h(w)$) gives the following result: let us define for $i \in Q_w$ its coheight $\text{coh}(i) = h(w) - h(i)$. Then we have

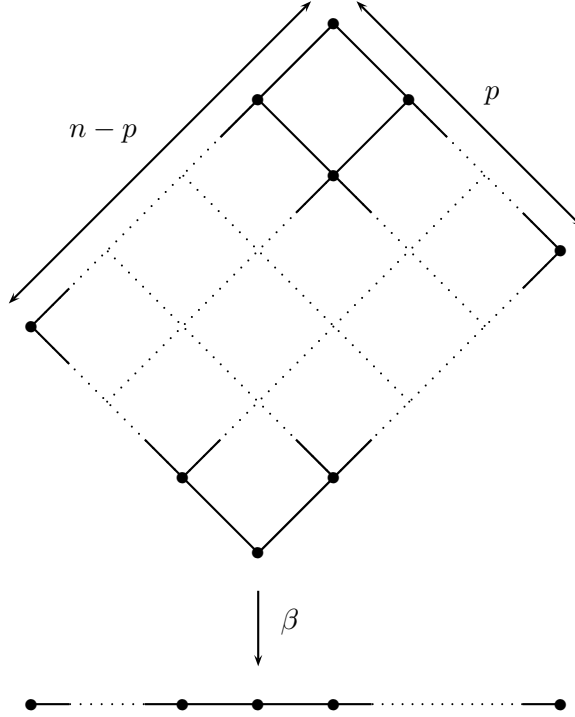
$$e_{\text{st}}(X(w)) = \prod_{i \in Q_w} \left(1 + \frac{1}{1 + \text{coh}(i)} \right).$$

8 Appendix

In this appendix we give the quivers of minuscule homogeneous varieties and describe the quivers of minuscule Schubert varieties.

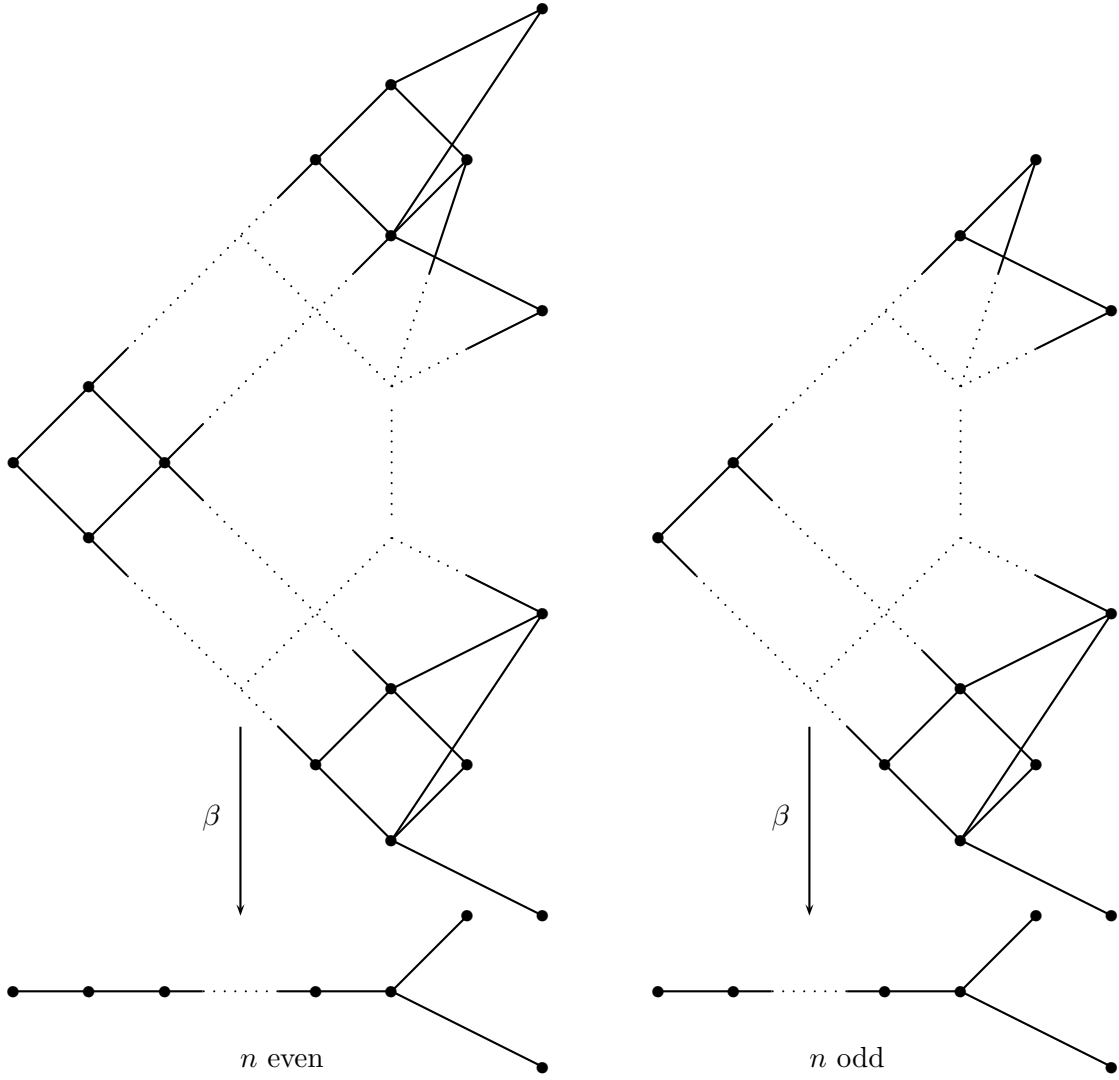
8.1 Quivers of minuscule homogeneous varieties

The following quiver is the quiver of the grassmannian of p -dimensional subvector spaces of an n -dimensional vector space. The morphism β associating to any vertex a simple root is simply the vertical projection on the Dynkin diagram.

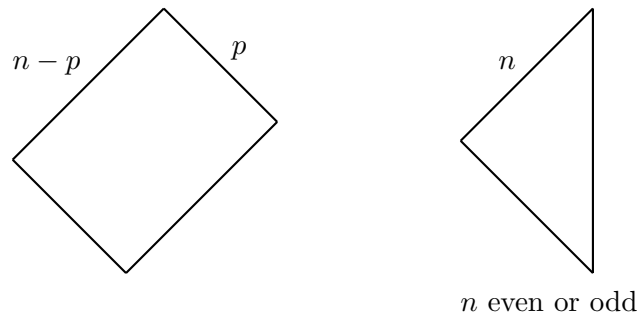


It is easy to verify that this diagram satisfies the geometric conditions of proposition 4.2 so that it corresponds to a Schubert variety of dimension $p(n-p)$ of the grassmannian. It must be the quiver of the grassmannian.

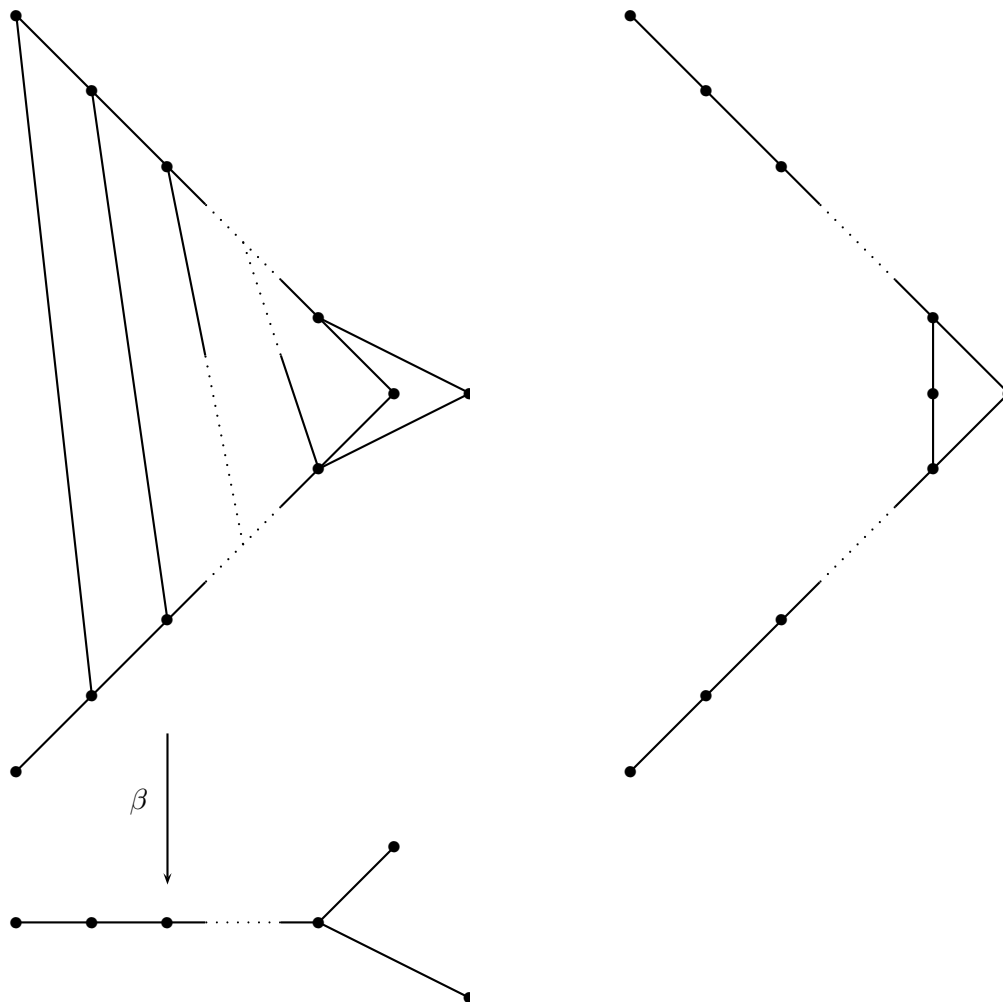
In the same way, the quiver of the grassmannian of maximal isotropic subspaces in a $2n$ -dimensional vector space endowed with a non degenerate quadratic form is given by (one more time, the morphism β is given by the vertical projection on the Dynkin diagram) one of the following form depending on the parity of n :



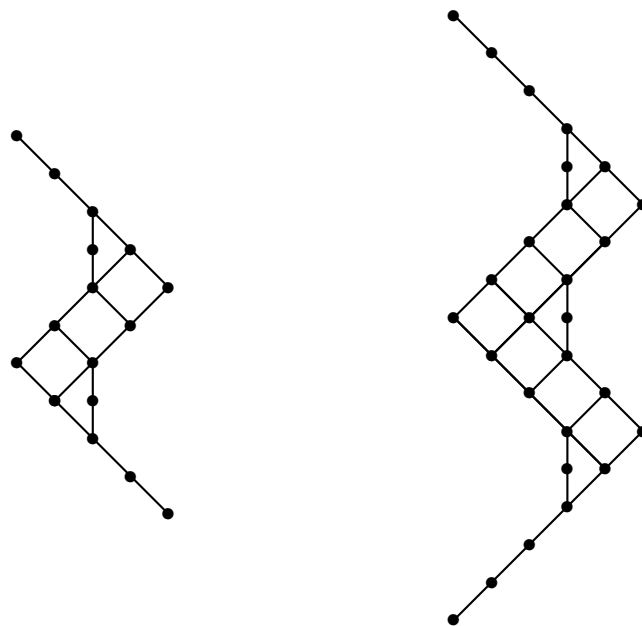
In the text we use for the quiver of the grassmannian and for the quiver of the grassmannian of maximal isotropic subspaces the following schematic version of the quivers:



For the even dimensional quadrics, we get the following quiver (we often draw it like the diagram on the right even if β is not exactly given by the projection and if we forget some arrows):

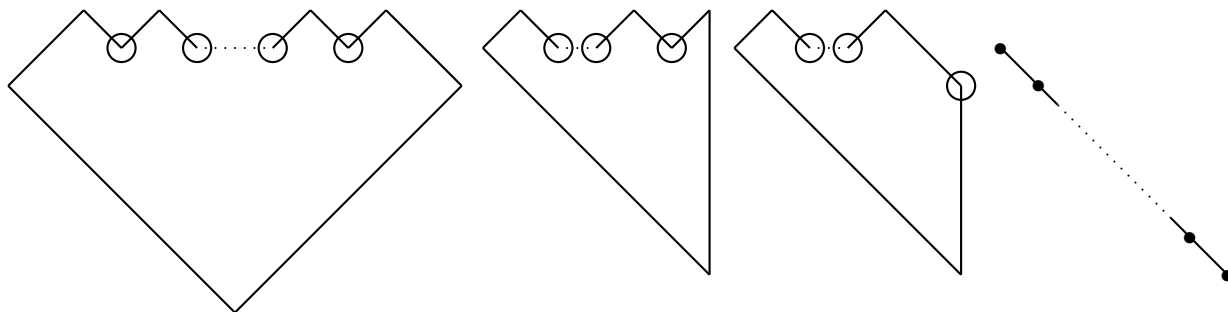


Finally for E_6 and E_7 we only draw the simplified versions where some arrows (easily detected) have been omitted and where the map β is not exactly the projection:

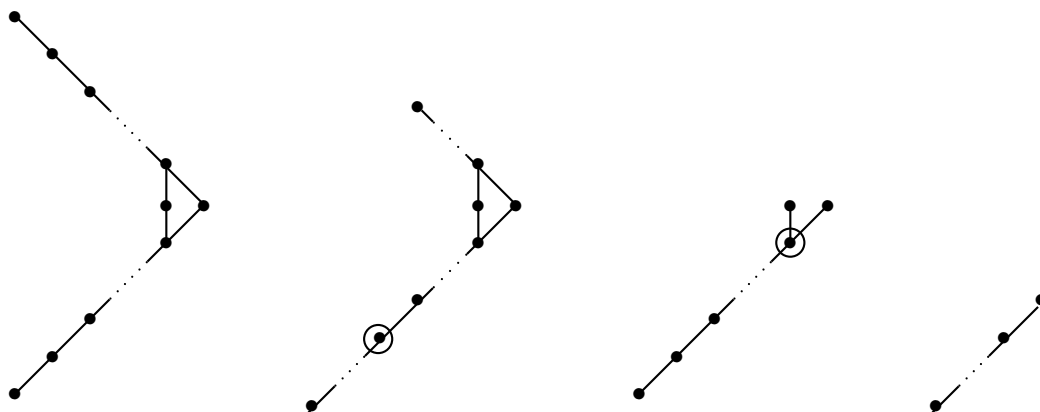


8.2 Quivers of minuscule Schubert varieties

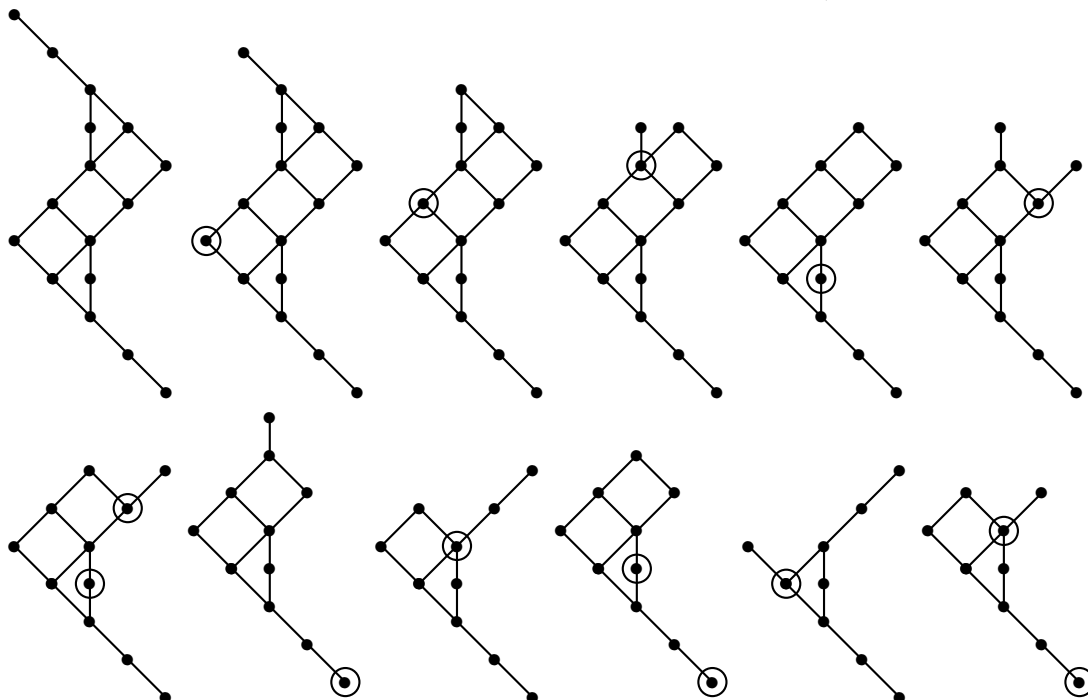
Thanks to the description of the quivers of minuscule homogeneous varieties and the proposition 4.5, we know that the quivers of a minuscule Schubert variety is of the following form (we have circled the successors of elements in the set A described in proposition 4.5):

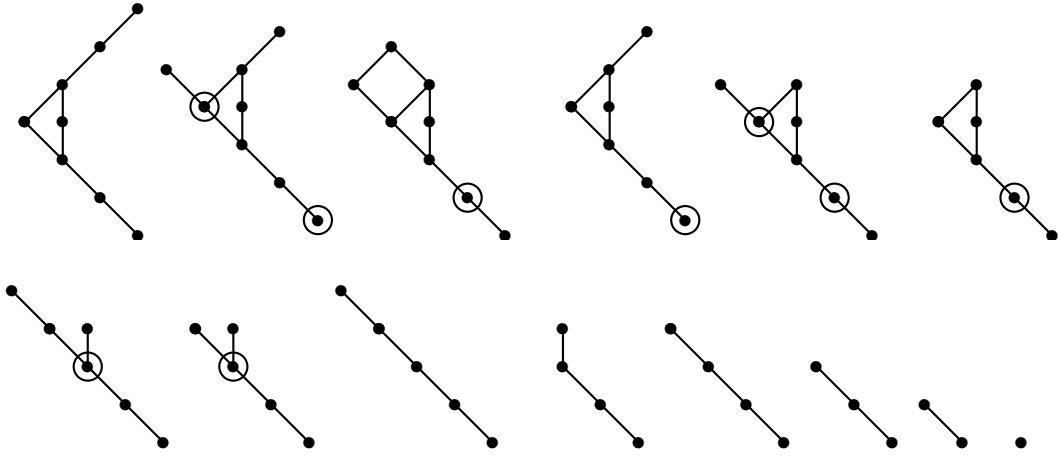


or for the quadrics:



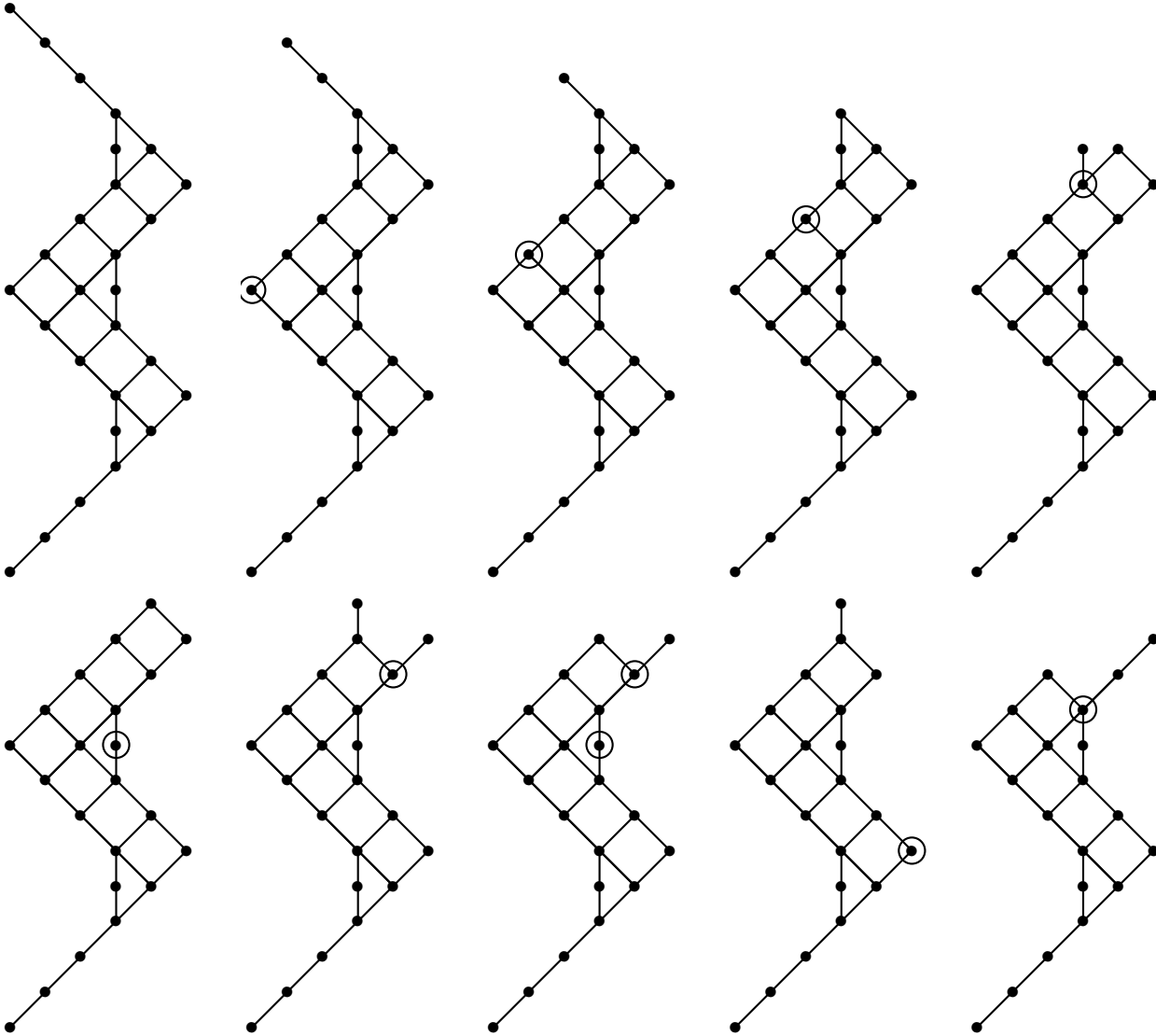
and finally for E_6 and E_7 let us give a complete list of the quivers (except the empty quiver):

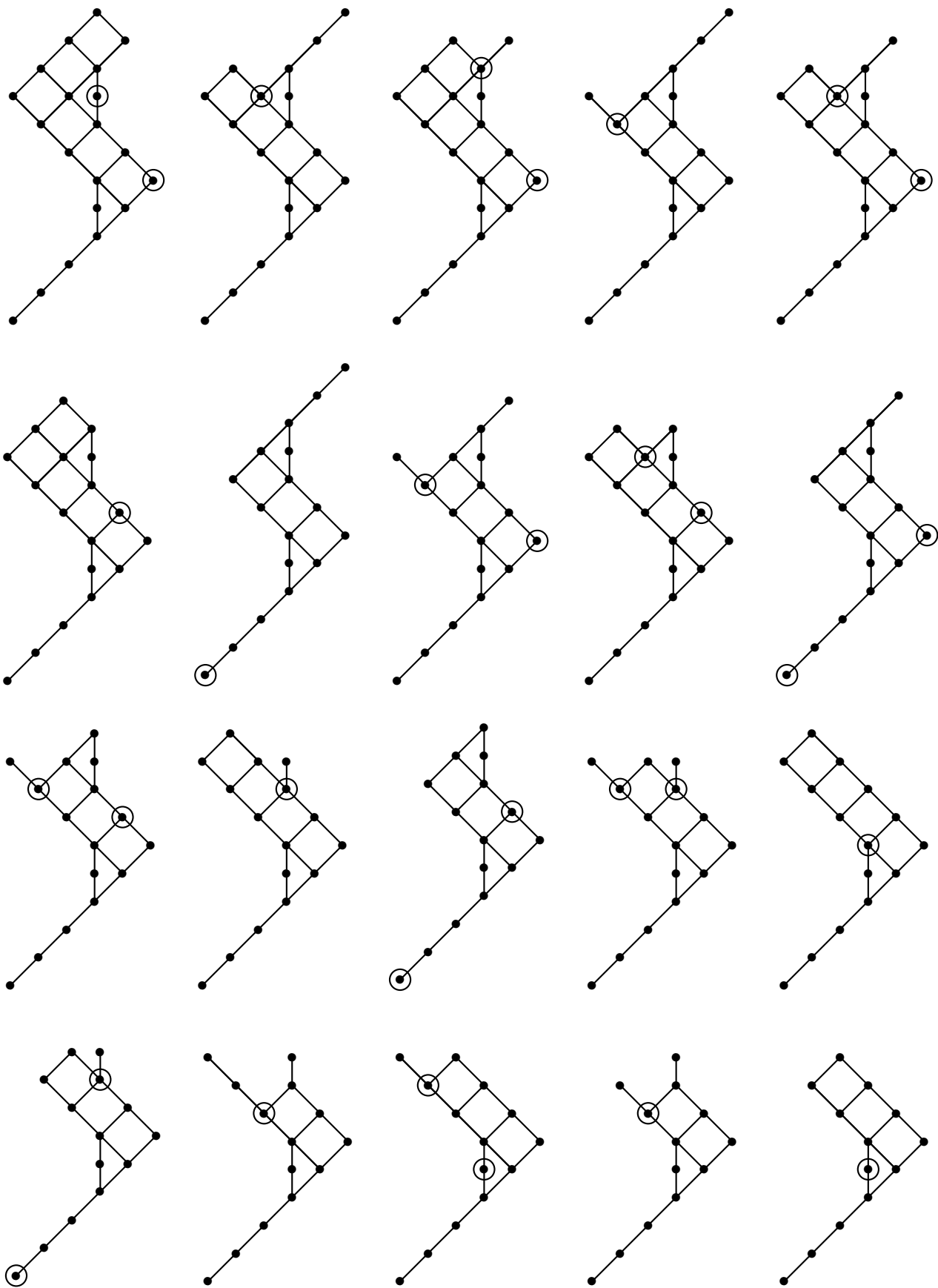


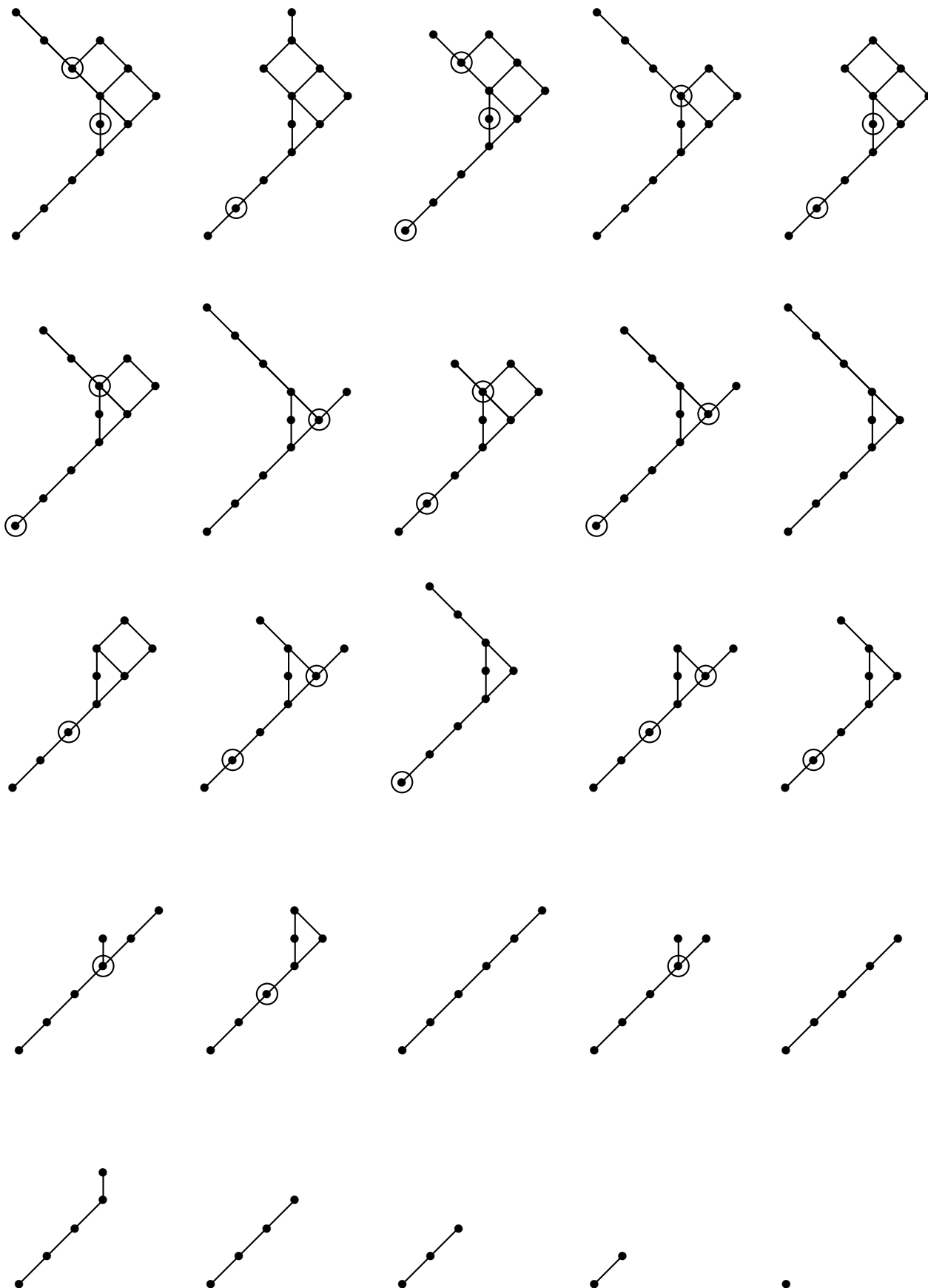


It is easy to verify (thanks to our results) that the only Schubert varieties admitting a IH -small resolution are the following (we only list there number in the previous list): 1, 6, 7, 9, 11, 13, 17, 19, 20, 21, 22, 23, 24, 25, 26 and the 0-dimensional one.

Let us now list the Schubert varieties for the E_7 case:







It is easy to verify (thanks to our results) that the only Schubert varieties admitting a IH -small resolution are the following (we only list there number in the previous list): 1, 24, 27, 28, 31, 34, 37, 40, 44, 46, 48, 49, 50, 51, 52, 53, 54, 55 and the 0-dimensional one.

References

- [Ba] *Victor, Batyrev*: Stringy Hodge numbers of varieties with Gorenstein canonical singularities. Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 1–32, World Sci. Publishing, River Edge, NJ, 1998.
- [BL] *Sara Billey and Venkatramani Lakshmibai*: Singular loci of Schubert varieties. Progress in Mathematics, 182. Birkhäuser Boston, Inc., Boston, MA, 2000.
- [BW] *Sara Billey and Gregory Warrington*: Maximal singular loci of Schubert varieties in $SL(n)/B$. Trans. Amer. Math. Soc. 355 (2003), no. 10, 3915–3945.
- [Bo] *Brian D. Boe*: Kazhdan-Lusztig polynomials for Hermitian symmetric spaces. Trans. Amer. Math. Soc. 309 (1988), no. 1, 279–294.
- [Bou] *Nicolas Bourbaki*: Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV : Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines. Actualités Scientifiques et Industrielles, No. 1337 Hermann, Paris 1968.
- [Br] *Michel Brion*: Lectures on the geometry of flag varieties. Topics in cohomological studies of algebraic varieties, 33–85, Trends Math., Birkhäuser, Basel, 2005.
- [BK] *Michel Brion and Shrawan Kumar*: Frobenius splitting methods in geometry and representation theory. Progress in Mathematics, 231. Birkhäuser Boston, Inc., Boston, MA, 2005.
- [BP] *Michel Brion and Patrick Polo*: Generic singularities of certain Schubert varieties. Math. Z. 231 (1999), no. 2, 301–324.
- [Co] *Carlos Contou Carrère*: Le lieu singulier des variétés de Schubert. Adv. in Math. 71 (1988), no. 2, 186–231.
- [Cor1] *Aurélien Cortez*: Singularités génériques des variétés de Schubert covexillaires. Ann. Inst. Fourier (Grenoble) 51 (2001), no. 2, 375–393.
- [Cor2] *Aurélien Cortez*: Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire. Adv. Math. 178 (2003), no. 2, 396–445.
- [De] *Michel Demazure*: Désingularisation des variétés de Schubert généralisées. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I. Ann. Sci. École Norm. Sup. (4) 7 (1974), 53–88.
- [FMPSS] *William Fulton, Robert MacPherson, Frank Sottile and Bernd Sturmfels*: Intersection theory on spherical varieties. J. Algebraic Geom. 4 (1995), no. 1.
- [Gr] *Alexander Grothendieck* : Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. Inst. Hautes Études Sci. Publ. Math. No. 32 (1967).
- [Ha] *Hans Christian Hansen*: On cycles on flag manifolds, Math. Scand. 33 (1973), 269–274.

- [KLR] *Christian Kassel, Alain Lascoux and Chritophe Reutenauer*: The singular locus of a Schubert variety. J. Algebra 269 (2003), no. 1, 74–108.
- [Ke] *George R. Kempf*: Vanishing theorems for flag manifolds. Amer. J. Math. 98 (1976), no. 2, 325–331.
- [LLM] *Venkatramani Lakshmibai, Peter Littelmann and Peter Magyar*: Standard monomial theory for Bott-Samelson varieties. Compositio Math. 130 (2002), no. 3, 293–318.
- [LMS] *Venkatramani Lakshmibai, Chitikila Musili and Conjeerveram S. Seshadri*: Geometry of G/P . III. Standard monomial theory for a quasi-minuscule P . Proc. Indian Acad. Sci. Sect. A Math. Sci. 88 (1979), no. 3, 93–177.
- [LS] *Alain Lascoux and Marcel-Paul Schützenberger*: Polynômes de Kazhdan et Lusztig pour les grassmanniennes. Young tableaux and Schur functors in algebra and geometry (Toruń, 1980), pp. 249–266, Astérisque, 87–88, Soc. Math. France, Paris, 1981.
- [LT] *Niels Lauritzen and Jesper Funch Thomsen* : Line bundles on Bott-Samelson varieties. J. Algebraic Geometry 13 (2004) 461–473.
- [Ma] *Peter Magyar*: Borel-Weil theorem for configuration varieties and Schur modules. Adv. Math. 134 (1998), no. 2, 328–366.
- [Man1] *Laurent Manivel*: Le lieu singulier des variétés de Schubert. Internat. Math. Res. Notices 2001, no. 16, 849–871.
- [Man2] *Laurent Manivel* : Generic singularities of Schubert Varieties. math.AG/0105239 (2001).
- [Mat] *Kenji Matsuki*: Introduction to the Mori program. Universitext. Springer-Verlag, New York, 2002. xxiv+478 pp.
- [MR] *Vikram B. Mehta and A. Ramanathan*: Frobenius splitting and cohomology vanishing for Schubert varieties. Ann. of Math. (2) 122 (1985), no. 1, 27–40.
- [Pe1] *Nicolas Perrin*: Rational curves on minuscule Schubert varieties. J. of Algebra 294 (2005), no. 2, 431–462.
- [Pe2] *Nicolas Perrin*: Géométrie de variétés de Schubert, in preparation.
- [Pe3] *Nicolas Perrin*: Une désingularisation plus fine des variétés de Schubert. Preprint: math.AG/0003199 (2000).
- [Pe4] *Nicolas Perrin*: Courbes elliptiques sur la variété spinorielle de dimension 10. Preprint math.AG/0409125.
- [Pe5] *Nicolas Perrin*: Rational curves on homogeneous cones. Doc. Math. 9 (2004), 623–637.
- [RR] *Sundararaman Ramanan and A. Ramanathan*: Projective normality of flag varieties and Schubert varieties. Invent. Math. 79 (1985), no. 2, 217–224.
- [Ra] *A. Ramanathan*: Schubert varieties are arithmetically Cohen-Macaulay. Invent. Math. 80 (1985), no. 2, 283–294.
- [SV1] *Parameswaran Sankaran and P. Vanchinathan*: Small resolutions of Schubert varieties in symplectic and orthogonal Grassmannians. Publ. Res. Inst. Math. Sci. 30 (1994), no. 3, 443–458.

- [SV2] *Parameswaran Sankaran and P. Vanchinathan*: Small resolutions of Schubert varieties and Kazhdan-Lusztig polynomials. Publ. Res. Inst. Math. Sci. 31 (1995), no. 3, 465–480.
- [St] *John R. Stembridge*: Some combinatorial aspects of reduced words in finite Coxeter groups. Trans. Amer. Math. Soc. 349 (1997), no. 4, 1285–1332.
- [To] *Burt Totaro*: Chern numbers for singular varieties and elliptic homology. Ann. of Math. (2) 151 (2000), no. 2, 757–791.
- [Wi] *Jarosław A. Wiśniewski* On contractions of extremal rays of Rano manifolds. J. Reine Angew. Math. 417 (1991), 141–157.
- [WY] *Alexander Woo and Alexander Yong* : When is a Schubert variety Gorenstein?, math.AG/0409490 (2004).
- [Ze] *Andrei Zelevinsky* : Small resolutions of singularities of Schubert varieties. Funktsional. Anal. i Prilozhen. 17 (1983), no. 2, 75–77.
- [Zel] *Shmuel Zelikson* : Auslander-Reiten quivers and the Coxeter complex. (English. English summary) Algebr. Represent. Theory 8 (2005), no. 1, 35–55.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU
 175 RUE DU CHEVALERET
 75013 PARIS, FRANCE.
 email : `nperrin@math.jussieu.fr`